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Centers of projective vector fields of spatial quasi-homogeneous systems with weight (m, m, n) and degree 2 on the sphere

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Abstract. In this paper we study the centers of projective vector fields $\mathbf{Q}_T$ of three-dimensional quasi-homogeneous differential system $d\mathbf{x}/dt = \mathbf{Q}(\mathbf{x})$ with the weight (m, m, n) and degree 2 on the unit sphere $\mathbb{S}^2$. We seek the sufficient and necessary conditions under which $\mathbf{Q}_T$ has at least one center on $\mathbb{S}^2$. Moreover, we provide the exact number and the positions of the centers of $\mathbf{Q}_T$. First we give the complete classification of systems $d\mathbf{x}/dt = \mathbf{Q}(\mathbf{x})$ and then, using the induced systems of $\mathbf{Q}_T$ on the local charts of $\mathbb{S}^2$, we determine the conditions for the existence of centers. The results of this paper provide a convenient criterion to find out all the centers of $\mathbf{Q}_T$ on $\mathbb{S}^2$ with $\mathbf{Q}$ being the quasi-homogeneous polynomial vector field of weight (m, m, n) and degree 2.

Keywords: projective vector field, quasi-homogeneous system, sufficient and necessary conditions for centers.

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1 Introduction

We consider the polynomial differential systems in $\mathbb{R}^3$

$$\frac{d\mathbf{x}}{dt} = \mathbf{Q}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{Q}(\mathbf{x}) = (Q_1(\mathbf{x}), Q_2(\mathbf{x}), Q_3(\mathbf{x}))$. System (1.1) is called a quasi-homogeneous polynomial differential system with weight $(\alpha_1, \alpha_2, \alpha_3)$ and degree d if $\mathbf{Q}(\mathbf{x})$ is a quasi-homogeneous polynomial vector field with weight $(\alpha_1, \alpha_2, \alpha_3)$ and degree d, i.e.,

$$Q_i(\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \lambda^{\alpha_3} x_3) = \lambda^{\alpha_i - 1 + d} Q_i(x_1, x_2, x_3), \qquad i = 1, 2, 3, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ and $d, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^+$. In particular, if $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$, then system (1.1) is a homogeneous polynomial system of degree d.

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The three-dimensional polynomial differential systems occur as models or at least as simplifications of models in many domains in science. For example, the population models in biology. In recent years, the qualitative theory of three-dimensional polynomial differential systems has been and still is receiving intensive attention [1,2,5,7,9,10,13,15].

Just as the author of [16] point out that, the study of three-dimensional polynomial differential system is much more difficult than that of planar polynomial system. For example, it is an arduous task to determine the global topological structure of the Lorenz system

$$\dot{x}_1 = \sigma(x_2 - x_1), \quad \dot{x}_2 = \rho x_1 - y - xz, \quad \dot{x}_3 = -\beta z + xy \quad (\sigma, \beta, \rho \geq 0),$$

although this system has a simply form, see [14].

An efficient method for studying the qualitative behavior of orbits of system (1.1) is to project the system to the unit sphere $\mathbb{S}^2$. In what follows we will adopt the notations used in [6] to introduce some basic theory of the projective system on $\mathbb{S}^2$.

By taking the transformation of coordinates

$$\mathbf{x} = (x_1, x_2, x_3) = (r^{\alpha_1} y_1, r^{\alpha_2} y_2, r^{\alpha_3} y_3), \qquad \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{S}^2, \qquad r \in \mathbb{R}^+,$$

we get from system (1.1) that

$$\frac{dr}{d\tau} = r\langle \mathbf{y}, \mathbf{Q}(\mathbf{y})\rangle =: r \cdot R(\mathbf{y}), \tag{1.3}$$

$$\frac{d\mathbf{y}}{d\tau} = \langle \bar{\mathbf{y}}, \mathbf{y}\rangle \mathbf{Q}(\mathbf{y}) - \langle \mathbf{y}, \mathbf{Q}(\mathbf{y})\rangle \bar{\mathbf{y}} =: \mathbf{Q}_T(\mathbf{y}), \tag{1.4}$$

where $\bar{\mathbf{y}} = (\alpha_1 y_1, \alpha_2 y_2, \alpha_3 y_3)$ and $d\tau = (r^{d-1}/\langle \bar{\mathbf{y}}, \mathbf{y}\rangle)\, dt$.

System (1.4) plays an important role in the analysis of the topology of system (1.1). Indeed, if we write Γ, g and C for trajectory, singularity and closed orbit of system (1.4) on the $\mathbb{S}^2$, respectively, and let $\mathbf{y} = \mathbf{y}(\tau, \mathbf{y}_0)$ be the expression of Γ (resp. g, C) with initial value $\mathbf{y}_0 = \mathbf{y}(\tau_0, \mathbf{y}_0)$, then $\mathbf{y}(\tau, \mathbf{y}_0)$ is defined on $\mathbb{R}$ and

$$r(\tau, \tau_0) = r_0 \exp \int_{\tau_0}^{\tau} R(\mathbf{y}(s, \mathbf{y}_0))\, ds$$

is the solution of (1.3). Hence we obtain the corresponding trajectory of system (1.1)

$$W_\Gamma \;\; (\text{resp. } W_g, W_C) = \{(r^{\alpha_1}(\tau,\tau_0) y_1(\tau, \mathbf{y}_0), r^{\alpha_2}(\tau,\tau_0) y_2(\tau, \mathbf{y}_0), r^{\alpha_3}(\tau,\tau_0) y_3(\tau, \mathbf{y}_0)) \mid \tau \in \mathbb{R}\}.$$

For any $\mathbf{y} \in \mathbb{S}^2$, we define a curve as $S(\mathbf{y}) = \{(r^{\alpha_1} y_1, r^{\alpha_2} y_2, r^{\alpha_3} y_3) \mid r > 0\}$. The orbit Γ of system (1.4) on $\mathbb{S}^2$ can be regarded as the projection of W_Γ along the family of curves $\{S(\mathbf{y}) \mid \mathbf{y} \in \mathbb{S}^2\}$. In this sense, we call $\mathbf{Q}_T(\mathbf{y})$ the *projective vector field* of $\mathbf{Q}(\mathbf{y})$ on $\mathbb{S}^2$ and call (1.4) *the projective system* of (1.1).

To study the behavior of orbits of system (1.4), we use the local charts of $\mathbb{S}^2$. Denoted by

$$H_i^+ = \{\mathbf{x} \in \mathbb{R}^3 : x_i > 0\}, \qquad H_i^- = \{\mathbf{x} \in \mathbb{R}^3 : x_i < 0\}$$

and

$$\Pi_i^+ = \{\bar{\mathbf{x}} \in \mathbb{R}^3 : \bar{x}_i = 1\}, \qquad \Pi_i^- = \{\bar{\mathbf{x}} \in \mathbb{R}^3 : \bar{x}_i = -1\}.$$

Define respectively the coordinate transformations $\phi_+^i : H_i^+ \cap \mathbb{S}^2 \to \Pi_i^+$ and $\phi_-^i : H_i^- \cap \mathbb{S}^2 \to \Pi_i^-$

$$\bar{\mathbf{x}} = \phi_+^i(\mathbf{y}) = \left(\frac{y_1}{y_i^{\alpha_1/\alpha_i}}, \frac{y_2}{y_i^{\alpha_2/\alpha_i}}, \frac{y_3}{y_i^{\alpha_3/\alpha_i}} \right)$$

and

$$\bar{\mathbf{x}} = \phi_-^i(\mathbf{y}) = \left(\frac{y_1}{|y_i|^{\alpha_1/\alpha_i}}, \frac{y_2}{|y_i|^{\alpha_2/\alpha_i}}, \frac{y_3}{|y_i|^{\alpha_3/\alpha_i}} \right),$$

for $i = 1,2,3$. It is easy to see that $\{(H_i^{\pm} \cap \mathbb{S}^2, \phi_{\pm}^i) : i = 1,2,3\}$ is the set of local charts of $\mathbb{S}^2$. System (1.4) in these local charts is topologically equivalent to

$$\frac{d\bar{\mathbf{x}}}{d\bar{\tau}} = W_+^i(\bar{\mathbf{x}}) = \left(Q_1(\bar{\mathbf{x}}) - \frac{\alpha_1}{\alpha_i}\bar{x}_1 Q_i(\bar{\mathbf{x}}), Q_2(\bar{\mathbf{x}}) - \frac{\alpha_2}{\alpha_i}\bar{x}_2 Q_i(\bar{\mathbf{x}}), Q_3(\bar{\mathbf{x}}) - \frac{\alpha_3}{\alpha_i}\bar{x}_3 Q_i(\bar{\mathbf{x}}) \right),$$
$$\frac{d\bar{\mathbf{x}}}{d\bar{\tau}} = W_-^i(\bar{\mathbf{x}}) = \left(Q_1(\bar{\mathbf{x}}) + \frac{\alpha_1}{\alpha_i}\bar{x}_1 Q_i(\bar{\mathbf{x}}), Q_2(\bar{\mathbf{x}}) + \frac{\alpha_2}{\alpha_i}\bar{x}_2 Q_i(\bar{\mathbf{x}}), Q_3(\bar{\mathbf{x}}) + \frac{\alpha_3}{\alpha_i}\bar{x}_3 Q_i(\bar{\mathbf{x}}) \right),$$

where $i = 1,2,3$ and $d\bar{\tau} = \langle \bar{\mathbf{y}}, \mathbf{y} \rangle |y_i|^{(d-1)/\alpha_i} d\tau$.

In the literature many authors study the projective vector field of system (1.1) with degree two ($d = 2$). Most of them consider the homogeneous case, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = 1$. For instance, Camacho in [1] investigates the projective vector fields of homogeneous polynomial system of degree two. The classification of projective vector fields without periodic orbits on $\mathbb{S}^2$ is given. Wu in [15] corrects some mistakes of [1] and provide several properties of homogeneous vector fields of degree two. Llibre and Pessoa in [10] study the homogeneous polynomial vector fields of degree two, it was shown that if the vector field on $\mathbb{S}^2$ has finitely many invariant circles, then every invariant circle is a great circle. [11] deals with the phase portraits for quadratic homogeneous polynomial vector fields on $\mathbb{S}^2$, they verify that if the vector field has at least a non-hyperbolic singularity, then it has no limit cycles. They also give necessary and sufficient conditions for determining whether a singularity of (1.4) on $\mathbb{S}^2$ is a center. Pereira and Pessoa in [12] classify all the centers of a certain class of quadratic reversible polynomial vector fields on $\mathbb{S}^2$.

Under the homogeneity assumption we know that whenever $\mathbf{x}(t)$ is a solution of system (1.1), then so is $\tilde{\mathbf{x}} = \lambda \mathbf{x}(\lambda^{d-1}t)$. This conclusion can be extended to quasi-homogeneous systems. Indeed, from the quasi-homogeneity, $\tilde{\mathbf{x}}(t) = \mathrm{diag}(\lambda^{\alpha_1}, \lambda^{\alpha_2}, \lambda^{\alpha_3})\mathbf{x}(\lambda^{d-1}t)$ is a solution of (1.1) when $\mathbf{x}(t)$ is a solution of (1.1). Recently, the authors of [6] study the projective vector field of a three-dimensional quasi-homogeneous system with weight $(1,1,\alpha)$, with $\alpha > 1$, and degree $d = 2$. Some interesting qualitative behaviours are determined according to the parameters of the systems. Another meaningful work about the spatial quasi-homogeneous systems is [7]. In that paper the authors generalize the results of [2,13] by studying the limit set of trajectories of three-dimensional quasi-homogeneous systems. They also point out, by a counterexample, the mistake of [2].

However, to the best of our knowledge, there is no paper dealing with the center of the projective vector field of spatial quasi-homogeneous systems. Motivated by this fact, in the present paper we study the sufficient and necessary conditions for the projective vector field $\mathbf{Q}_T$ of the system (1.1) with the weight (m,m,n) and degree 2 to have at least one center on $\mathbb{S}^2$. We would like to emphasize that, in the above mentioned papers dealing with homogeneous systems, many authors concern on the periodic orbits of system (1.4), see [1,6,11]. This is because the periodic behavior of system (1.4) provide a threshold to investigate the periodic and spirally behaviors of the spatial system. Our work provides a criterion for the projective vector field associated to system (1.1) to have a family of periodic orbits.

This paper is organized as follows. In Section 2, we prove some properties and establish the canonical forms of quasi-homogeneous polynomial system (1.1) with weight (m,m,n) and degree 2. In Sections 3, 4, and 5, we are going to seek the sufficient and necessary conditions under which the projective system (1.4) has at least one center on $\mathbb{S}^2$, where Section 3 (resp. Section 4 and Section 5) deals with the case that $n = 1$ (resp. $m > 1, n > 1$ and $m = 1$).

2 Properties and canonical forms for quasi-homogeneous systems with weight (m, m, n) and degree $d = 2$

The first goal of this section is to derive some properties of the three-dimensional quasi-homogeneous polynomial vector field with weight (m, m, n) and degree $d = 2$. The results obtained will be used in the next sections.

Define a homomorphism $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ as $\psi(x_1, x_2, x_3) = ((-1)^m x_1, (-1)^m x_2, (-1)^n x_3)$.

Proposition 2.1. *Assume that $\mathbf{Q}$ is a quasi-homogeneous polynomial vector field with weight (m, m, n) and degree $d = 2$. Then*

$$\mathbf{Q}(\psi(\mathbf{x})) = -D\psi \cdot \mathbf{Q}(\mathbf{x}), \qquad \mathbf{Q}_T(\psi(\mathbf{y})) = -D\psi \cdot \mathbf{Q}_T(\mathbf{y}).$$

Proof. Firstly, it follows from the quasi-homogeneity property of $\mathbf{Q}$ that

$$\begin{aligned}\mathbf{Q}(\psi(\mathbf{x})) &= \mathbf{Q}((-1)^m x_1, (-1)^m x_2, (-1)^n x_3)\\ &= ((-1)^{m+1} Q_1(\mathbf{x}), (-1)^{m+1} Q_2(\mathbf{x}), (-1)^{n+1} Q_3(\mathbf{x}))\\ &= -D\psi \cdot Q(\mathbf{x}).\end{aligned}$$

Secondly, by the expression of $\mathbf{Q}_T(\mathbf{y})$ we have

$$\mathbf{Q}_T(\psi(\mathbf{y})) = \langle \overline{\psi(\mathbf{y})}, \psi(\mathbf{y}) \rangle \mathbf{Q}(\psi(\mathbf{y})) - \langle \psi(\mathbf{y}), \mathbf{Q}(\psi(\mathbf{y})) \rangle \overline{\psi(\mathbf{y})}.$$

Since $\overline{\psi(\mathbf{y})} = D\psi \cdot \bar{\mathbf{y}}$, it follows that $\langle \overline{\psi(\mathbf{y})}, \psi(\mathbf{y}) \rangle = \langle \bar{\mathbf{y}}, \mathbf{y} \rangle$. Hence

$$\begin{aligned}\mathbf{Q}_T(\psi(\mathbf{y})) &= -\langle \bar{\mathbf{y}}, \mathbf{y} \rangle D\psi \cdot Q(\mathbf{y}) + \langle \psi(\mathbf{y}), D\psi \cdot \mathbf{Q}(\mathbf{y}) \rangle \overline{\psi(\mathbf{y})}\\ &= -\langle \bar{\mathbf{y}}, \mathbf{y} \rangle D\psi \cdot \mathbf{Q}(\mathbf{y}) + \langle \mathbf{y}, \mathbf{Q}(\mathbf{y}) \rangle \overline{\psi(\mathbf{y})}\\ &= -D\psi \cdot (\langle \bar{\mathbf{y}}, \mathbf{y} \rangle Q(\mathbf{y}) - \langle \mathbf{y}, \mathbf{Q}(\mathbf{y}) \rangle \bar{\mathbf{y}}) = -D\psi \cdot \mathbf{Q}_T(\mathbf{y}).\end{aligned}$$

The proof is finished. □

Proposition 2.2. *Assume that $\mathbf{Q}$ is a quasi-homogeneous polynomial vector field with weight (m, m, n). Let $L = \{(\lambda \cos \alpha_0, \lambda \sin \alpha_0, 1) \mid \lambda \in \mathbb{R}\}$ be a straight line on Π_3^+. If $S \subset \mathbb{S}^2$ is a great circle which contains the points $(0, 0, \pm 1)$ and $(\cos \alpha_0, \sin \alpha_0, 0)$, then $(\phi_+^3)^{-1}(L) = H_3^+ \cap S$.*

Proof. Since S is a great circle containing the points $(0, 0, \pm 1)$ and $(\cos \alpha_0, \sin \alpha_0, 0)$, we find

$$H_3^+ \cap S = \{(\pm \cos \alpha_0 \sin \theta, \pm \sin \alpha_0 \sin \theta, \cos \theta) \mid \theta \in [0, \pi/2)\}.$$

Hence

$$\begin{aligned}\phi_+^3(H_3^+ \cap S) &= \left\{ \left(\frac{y_1}{(y_3)^{m/n}}, \frac{y_2}{(y_3)^{m/n}}, 1 \right) \middle| (y_1, y_2, y_3) = (\pm \cos \alpha_0 \sin \theta, \pm \sin \alpha_0 \sin \theta, \cos \theta), \right.\\ &\qquad \left. \theta \in [0, \pi/2) \right\}\\ &= \left\{ (\lambda \cos \alpha_0, \lambda \sin \alpha_0, 1) \mid \lambda = \pm \sin \theta / (\cos \theta)^{m/n}, \theta \in [0, \pi/2) \right\}\\ &= \{(\lambda \cos \alpha_0, \lambda \sin \alpha_0, 1) \mid \lambda \in (-\infty, \infty)\}.\end{aligned}$$

The proof finishes because the above expression is equivalent to $(\phi_+^3)^{-1}(L) = H_3^+ \cap S$. □

The second purpose of this section is to obtain the canonical form for the quasi-homogeneous polynomial vector fields with weight (m, m, n) and degree $d = 2$, where m and n are two different positive integers.

Lemma 2.3. *Every three-dimensional quasi-homogeneous polynomial differential system (1.1) with weight (m, m, n) and degree $d = 2$ can be written in one of the following forms:*

(1) If $m = 1$, then

$$\begin{aligned} \frac{dx_1}{dt} &= \sum_{i+j=2} a_{i,j,0}x_1^i x_2^j + (1-\operatorname{sgn}(n-2))a_{0,0,1}x_3, \\ \frac{dx_2}{dt} &= \sum_{i+j=2} b_{i,j,0}x_1^i x_2^j + (1-\operatorname{sgn}(n-2))b_{0,0,1}x_3, \\ \frac{dx_3}{dt} &= a_{1,0,1}x_1x_3 + a_{0,1,1}x_2x_3 + \sum_{i+j=n+1} c_{i,j,0}x_1^i x_2^j. \end{aligned} \tag{2.1}$$

(2) If $n = 1$, then

$$\begin{aligned} \frac{dx_1}{dt} &= a_{1,0,1}x_1x_3 + a_{0,1,1}x_2x_3 + a_{0,0,m+1}x_3^{m+1}, \\ \frac{dx_2}{dt} &= b_{1,0,1}x_1x_3 + b_{0,1,1}x_2x_3 + b_{0,0,m+1}x_3^{m+1}, \\ \frac{dx_3}{dt} &= (1-\operatorname{sgn}(m-2))(c_{1,0,0}x_1 + c_{0,1,0}x_2) + c_{0,0,2}x_3^2. \end{aligned} \tag{2.2}$$

(3) If $m \geq 2$ and $n \geq 2$, then

$$\frac{dx_1}{dt} = a_{0,0,\frac{m+1}{n}}x_3^{\frac{m+1}{n}}, \qquad \frac{dx_2}{dt} = b_{0,0,\frac{m+1}{n}}x_3^{\frac{m+1}{n}}, \qquad \frac{dx_3}{dt} = \sum_{i+j=\frac{n+1}{m}} c_{i,j,0}x_1^i x_2^j. \tag{2.3}$$

Here $\operatorname{sgn}(\cdot)$ is the sign function and in (2.3) we define $a_{0,0,\frac{m+1}{n}} = b_{0,0,\frac{m+1}{n}} = 0$ if $(m+1)/n \notin \mathbb{N}^+$, and $c_{i,j,0} = 0$ if $i + j = (n+1)/m \notin \mathbb{N}^+$.

Proof. It follows from (1.2) that $Q_i(0,0,0) = (0,0,0)$. Thus we set

$$Q_i(x_1, x_2, x_3) = \sum_{k=1}^{n_i} q_{ik}(x_1, x_2, x_3),$$

where

$$q_{ik}(x_1, x_2, x_3) = \sum_{k_1+k_2=0}^{k} a^{(i)}_{k_1,k_2,k-k_1-k_2} x_1^{k_1} x_2^{k_2} x_3^{k-k_1-k_2}, \qquad i = 1, 2, 3.$$

Substituting the above expressions into (1.2) with $(\alpha_1, \alpha_2, \alpha_3) = (m, m, n)$ and $d = 2$ yields

$$a^{(i)}_{k_1,k_2,k-k_1-k_2}(\lambda^{m(k_1+k_2-1)+n(k-k_1-k_2)-1} - 1) = 0, \qquad k = 1, 2, \ldots, n_i,\ i = 1, 2, \tag{2.4}$$

$$a^{(3)}_{k_1,k_2,k-k_1-k_2}(\lambda^{m(k_1+k_2)+n(k-k_1-k_2-1)-1} - 1) = 0, \qquad k = 1, 2, \ldots, n_3. \tag{2.5}$$

We will apply (2.4) and (2.5) to find out all the coefficients which vanish.

Case 1. $m = 1,\ n \geq 2$. If $a^{(i)}_{k_1,k_2,k-k_1-k_2} \neq 0,\ i = 1,2$, then by (2.4) we have

$$k_1 + k_2 + n(k - k_1 - k_2) - 2 = 0. \tag{2.6}$$

Noting that $0 \leq k_1 + k_2 \leq k$, we deduce from (2.6) that $k_1 + k_2 = 0,\ k = 1,\ n = 2$ or $k_1 + k_2 = k = 2$. This prove the first and the second equation of (2.1).

If $a^{(3)}_{k_1,k_2,k-k_1-k_2} \neq 0$, then by (2.5)

$$k_1 + k_2 + n(k - k_1 - k_2 - 1) - 1 = 0. \tag{2.7}$$

The equation (2.7) is satisfied if and only if $k_1 + k_2 = 1,\ k = 2$ or $k_1 + k_2 = k = n + 1 \geq 3$. This proves the third equation of (2.1).

Case 2. $n = 1,\ m \geq 2$. If $a^{(i)}_{k_1,k_2,k-k_1-k_2} \neq 0,\ i = 1,2$, then we get

$$m(k_1 + k_2 - 1) + k - k_1 - k_2 - 1 = 0. \tag{2.8}$$

We deduce from (2.8) that $k_1 + k_2 = 0,\ k = m + 1 \geq 3$ or $k_1 + k_2 = 1,\ k = 2$. This proves the first and the second equation of (2.2).

If $a^{(3)}_{k_1,k_2,k-k_1-k_2} \neq 0$, then

$$m(k_1 + k_2) + k - k_1 - k_2 - 2 = 0. \tag{2.9}$$

The equation (2.9) is satisfied if and only if $k_1 + k_2 = 0,\ k = 2$ or $k_1 + k_2 = k = 1,\ m = 2$. This proves the third equation of (2.2).

Case 3. $n \geq 2,\ m \geq 2$. If $a^{(i)}_{k_1,k_2,k-k_1-k_2} \neq 0,\ i = 1,2$, then we have

$$m(k_1 + k_2 - 1) + n(k - k_1 - k_2) - 1 = 0. \tag{2.10}$$

Since $m(k_1 + k_2 - 1) = 1 - n(k - k_1 - k_2) \leq 1$, it is enough to consider two cases: $k_1 + k_2 = 0$ and $k_1 + k_2 = 1 \leq k$. Furthermore, $k_1 + k_2 = 1 \leq k$ is impossible because $n(k - 1) \neq 1$ for all $n \geq 2$. If $k_1 + k_2 = 0$, then we get from (2.10) that $kn = m + 1$. This means that Q_1 and Q_2 are two nonzero functions if and only if $n|(m + 1)$. And hence the first and the second equation of (2.3) are obtained.

If $a^{(3)}_{k_1,k_2,k-k_1-k_2} \neq 0$, then $m(k_1 + k_2) + n(k - k_1 - k_2 - 1) - 1 = 0$. This equality holds if and only if $k_1 + k_2 = k \geq 1,\ km = n + 1$. Thus we obtain the third equation of (2.3). □

From the above lemma we get next result.

Theorem 2.4. *Suppose that Q_i $(i = 1,2,3)$ of system (1.1) are nonzero functions. Then every quasi-homogeneous polynomial vector field (1.1) with weight (m,m,n) and degree $d = 2$ can be changed, under a suitable affine transformation, to:*

(i) System

$$\begin{aligned} \frac{dx_1}{dt} &= a_1x_1^2 + a_2x_1x_2 + a_3x_2^2, \\ \frac{dx_2}{dt} &= b_1x_1^2 + b_2x_1x_2 + b_3x_2^2, \\ \frac{dx_3}{dt} &= c_1x_1x_3 + c_2x_2x_3 + \sum_{i=0}^{n+1} d_ix_1^ix_2^{n+1-i}, \end{aligned} \tag{2.11}$$

with weight $(1,1,n)$, where $n \geq 3$, or *(ii)* *system*

$$\begin{aligned}\frac{dx_1}{dt} &= a_1x_1^2 + a_2x_1x_2 + a_3x_2^2,\\ \frac{dx_2}{dt} &= b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + x_3,\\ \frac{dx_3}{dt} &= c_1x_1x_3 + c_2x_2x_3 + \sum_{i=0}^{3} d_i x_1^i x_2^{3-i},\end{aligned} \tag{2.12}$$

with weight $(1,1,2)$, *or*

(iii) *system*

$$\begin{aligned}\frac{dx_1}{dt} &= a_1x_1x_3 + a_2x_2x_3,\\ \frac{dx_2}{dt} &= b_1x_1x_3 + b_2x_2x_3 + \eta x_3^3,\\ \frac{dx_3}{dt} &= c_1x_1 + c_2x_2 + c_0x_3^2,\end{aligned} \tag{2.13}$$

with weight $(2,2,1)$, *where* $\eta = 0, 1$, *or*

(iv) *system*

$$\begin{aligned}\frac{dx_1}{dt} &= a_1x_1x_3 + a_2x_2x_3,\\ \frac{dx_2}{dt} &= b_1x_1x_3 + b_2x_2x_3 + \eta x_3^{m+1},\\ \frac{dx_3}{dt} &= x_3^2,\end{aligned} \tag{2.14}$$

with weight $(m,m,1)$, *where* $\eta = 0, 1$, *and* $m \geq 3$, *or*

(v) *system*

$$\frac{dx_1}{dt} = x_3, \qquad \frac{dx_2}{dt} = x_3, \qquad \frac{dx_3}{dt} = c_1x_1^2 + c_2x_1x_2 + c_3x_2^2, \tag{2.15}$$

with weight $(2,2,3)$, *or*

(vi) *system*

$$\frac{dx_1}{dt} = x_3^2, \qquad \frac{dx_2}{dt} = x_3^2, \qquad \frac{dx_3}{dt} = c_1x_1 + c_2x_2, \tag{2.16}$$

with weight $(3,3,2)$.

Proof. Firstly, in the case that $m = 1$, the canonical forms (2.11) and (2.12) follow from [6] directly.

Let us consider the case that $n = 1$ and $m = 2$. If $a_{0,0,3} = b_{0,0,3} = 0$, then system (2.2) becomes (2.13) with $\eta = 0$. If $a_{0,0,3}^2 + b_{0,0,3}^2 \neq 0$, then by noting that system (2.2) has the same form under the change of variables $(x_1, x_2) \to (x_2, x_1)$, we can assume without loss of generality that $b_{0,0,3} \neq 0$. Taking the transformation $\mathbf{z} = (b_{0,0,3}x_1 - a_{0,0,3}x_2, b_{0,0,3}^{-1}x_2, x_3)$, and then using the symbol $\mathbf{x}$ instead of $\mathbf{z}$, we can change system (2.2) to (2.13) with $\eta = 1$. The case $n = 1$ and $m \geq 3$ can be dealt with in a similar way.

Finally, consider the case that $m \geq 2, n \geq 2$. Since Q_i $(i = 1, 2, 3)$ is nonzero function, it follows that $(m+1)/n$, $(n+1)/m \in \mathbb{N}^+$. Thus $m = n - 1$ or $m = n + 1$. If $m = n - 1$, we get from $n|(m+1)$ and $m|(n+1)$ that $n = 3$, $m = 2$. If $m = n+1$, then $n = 2$, $m = 3$. Consequently, we obtain

$$\frac{dx_1}{dt} = ax_3, \qquad \frac{dx_2}{dt} = bx_3, \qquad \frac{dx_3}{dt} = c_1x_1^2 + c_2x_1x_2 + c_3x_2^2, \qquad ab \neq 0$$

with weight $(2,2,3)$ and

$$\frac{dx_1}{dt} = ax_3^2, \qquad \frac{dx_2}{dt} = bx_3^2, \qquad \frac{dx_3}{dt} = c_1x_1 + c_2x_2, \qquad ab \neq 0$$

with weight $(3,3,2)$. By taking an affine transformation of variables, we get systems (2.15) and (2.16). □

The systems (2.11) and (2.12) are considered in [6]. It is shown that the projective system of system (2.11) has no closed orbits on $\mathbb{S}^2$. But the authors do not give the conditions for projective systems of system (2.12) to acquire at least one center. The purpose of the rest of this paper is to find the sufficient and necessary conditions for all the projective systems (1.4) of the systems in Theorem 2.4 to have at least one center.

3 Center of the quasi-homogeneous systems with weight $(m, m, 1)$

In this section we deal with the canonical forms of (2.13) and (2.14). The main results of this section are the following two theorems.

Theorem 3.1. *Suppose that $\mathbf{Q}_T$ is the projective vector field of system (2.13), then the following statements hold.*

(A) For $\eta = 0$, $\mathbf{Q}_T$ has at least one center if and only if one of the following two conditions is satisfied:

(1) $a_1 + b_2 = 4c_0$, $(b_2 - a_1)^2 + 4a_2b_1 < 0$;

(2) $J(c_1, c_2) = a_2c_1^2 + (b_2 - a_1)c_1c_2 - b_1c_2^2 \neq 0$.

In addition,

(i) If (1) is satisfied and $c_1^2 + c_2^2 \neq 0$, then $\mathbf{Q}_T$ has exactly three centers respectively at $(0,0,1)$, $(0,0,-1)$ and $E\left(c_2/\sqrt{c_1^2+c_2^2},\ -c_1/\sqrt{c_1^2+c_2^2}, 0\right)$ when $J(c_1,c_2) > 0$ or $-E$ when $J(c_1,c_2) < 0$.

(ii) If (1) is satisfied and $c_1 = c_2 = 0$, then $\mathbf{Q}_T$ has exactly two centers at $(0,0,1)$ and $(0,0,-1)$, respectively.

(iii) If (2) is satisfied but (1) is not satisfied, then $\mathbf{Q}_T$ has a unique center at E when $J(c_1,c_2) > 0$ or at $-E$ when $J(c_1,c_2) < 0$.

(B) For $\eta = 1$, $\mathbf{Q}_T$ has at least one center if and only if $J(c_1,c_2) \neq 0$ or one of following conditions is satisfied:

(1) $a_2 = 0$, $\bar{b}_2 = 2\bar{a}_1$, $\bar{a}_1c_1 - b_1c_2 \neq 0$, $\bar{a}_1^2 + 2c_2 < 0$;

(2) $a_2 \neq 0$, $\bar{b}_2 = -\bar{a}_1$, $c_1a_2 = \bar{a}_1c_2$, $(\bar{a}_1^4 + 2\bar{a}_1^2a_2b_1 + a_2^2b_1^2 + 2a_2b_1c_2)(a_2b_1 + \bar{a}_1^2) < 0$;

(3) $a_2 \neq 0$, $2\bar{a}_1^3 - 3\bar{a}_1^2\bar{b}_2 + 9\bar{a}_1a_2b_1 - 3\bar{a}_1\bar{b}_2^2 - 36\bar{a}_1c_2 + v9a_2b_1\bar{b}_2 + 2\bar{b}_2^3 + 54a_2c_1 + 18\bar{b}_2c_2 = 0$, $(9\bar{a}_1^2a_2c_1 + 27a_2^2b_1c_1 - 9\bar{a}_1a_2\bar{b}_2c_1 + 9a_2\bar{b}_2^2c_1 - 8\bar{a}_1^3c_2 - 27\bar{a}_1a_2b_1c_2 + 12\bar{a}_1^2\bar{b}_2c_2 - 6\bar{a}_1\bar{b}_2^2c_2 + \bar{b}_2^3c_2) \cdot (3a_2c_1 + \bar{b}_2c_2 - 2\bar{a}_1c_2) < 0$,

with $\bar{a}_1 = a_1 - 2c_0, \bar{b}_2 = b_2 - 2c_0$.

Moreover, if $J(c_1, c_2) > 0$ *(resp.* $J(c_1, c_2) < 0$*), then on the equator* $\mathbf{Q}_T$ *has a unique center at* E *(resp.* $-E$*). If the condition* (i) $(i \in \{1, 2, 3\})$ *of (B) holds, then* $\mathbf{Q}_T$ *has a unique center at* $\mathbf{y}_i$ *on* $\mathbb{S}^2 \cap H_3^+$ *and it has a unique center at* $-\mathbf{y}_i$ *on* $\mathbb{S}^2 \cap H_3^-$ *which satisfies* $\phi_+^3(\mathbf{y}_i) = (x_i^*, y_i^*, 1)$*, where*

$$\begin{aligned} x_1^* &= \frac{\bar{a}_1^2 + 2c_2}{2\bar{a}_1 c_1 - 2b_1 c_2}, & y_1^* &= -\frac{\bar{a}_1 b_1 + 2c_1}{2\bar{a}_1 c_1 - 2b_1 c_2}, \\ x_2^* &= -\frac{a_2}{a_2 b_1 + \bar{a}_1^2}, & y_2^* &= \frac{\bar{a}_1}{a_2 b_1 + \bar{a}_1^2}, \\ x_3^* &= \frac{a_2(\bar{a}_1 + \bar{b}_2)}{2(3a_2 c_1 + \bar{b}_2 c_2 - 2\bar{a}_1 c_2)}, & y_3^* &= \frac{(\bar{b}_2 - 2\bar{a}_1)(\bar{b}_2 + \bar{a}_1)}{6(3a_2 c_1 + \bar{b}_2 c_2 - 2\bar{a}_1 c_2)}. \end{aligned} \tag{3.1}$$

Theorem 3.2. *The projective vector field* $\mathbf{Q}_T$ *of system* (2.14) *with* $m > 2$ *has at least one center on* $\mathbb{S}^2$ *if and only if*

$$a_1 + b_2 = 2m, \qquad \Delta_1 = (b_2 - a_1)^2 + 4a_2 b_1 < 0. \tag{3.2}$$

Furthermore, if (3.2) *is satisfied then* $\mathbf{Q}_T$ *has exactly two centers at the points*

$$\left(\frac{\eta a_2 \lambda_0^m}{\Delta_1}, \frac{\eta (m - a_1) \lambda_0^m}{\Delta_1}, \eta\lambda_0 + 1 - \eta \right) \quad \text{and} \quad \left(\frac{\eta a_2 (-\lambda_0)^m}{\Delta_1}, \frac{\eta (m - a_1)(-\lambda_0)^m}{\Delta_1}, -\eta\lambda_0 + \eta - 1 \right),$$

where $\lambda = \lambda_0$ *is the unique positive solution of equation*

$$(a_2^2 + (m - a_1)^2)\lambda^{2m} + \Delta_1^2 \lambda^2 = \Delta_1^2. \tag{3.3}$$

3.1 Quasi-homogeneous systems with weight $(2, 2, 1)$

In this subsection we assume that $\mathbf{Q}_T$ is the projective vector field of system (2.13). We will firstly study the centers on the $H_3^+ \cap \mathbb{S}^2$. The centers on $H_3^- \cap \mathbb{S}^2$ can be obtained by the symmetry (see Proposition 2.1). By straightforward calculations we find that the induced system of $\mathbf{Q}_T$ on Π_3^+ is

$$\begin{aligned} \frac{d\bar{x}_1}{d\bar{\tau}} &= Q_1(\bar{\mathbf{x}}) - 2\bar{x}_1 Q_3(\bar{\mathbf{x}}) = (a_1 - 2c_0)\bar{x}_1 + a_2 \bar{x}_2 - 2c_1 \bar{x}_1^2 - 2c_2 \bar{x}_1 \bar{x}_2, \\ \frac{d\bar{x}_2}{d\bar{\tau}} &= Q_2(\bar{\mathbf{x}}) - 2\bar{x}_2 Q_3(\bar{\mathbf{x}}) = \eta + b_1 \bar{x}_1 + (b_2 - 2c_0)\bar{x}_2 - 2c_1 \bar{x}_1 \bar{x}_2 - 2c_2 \bar{x}_2^2, \qquad \eta = 0, 1. \end{aligned} \tag{3.4}$$

Proposition 3.3. *Assume that* $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{S}^2 \cap H_3^+$ *with* $p_3 \neq 0, \pm 1$ *is a singularity of* $\mathbf{Q}_T$*. If* $\eta = 0$*, then* $\mathbf{p}$ *is not a center of* $\mathbf{Q}_T$*.*

Proof. Since $\mathbf{Q}_T(\mathbf{p}) = 0$, it follows that

$$\langle \mathbf{q}, \mathbf{p} \rangle Q_i(\mathbf{p}) - 2\langle \mathbf{p}, \mathbf{Q}(\mathbf{p}) \rangle p_i = 0, \qquad i = 1, 2,$$

where $\mathbf{q} = (2p_1, 2p_2, p_3)$. Thus

$$0 = p_1(Q_2 + 2p_2 Q_3) - p_2(Q_1 + 2p_1 Q_3) = p_1 Q_2 - p_2 Q_1 = p_3[I(p_1, p_2) + \eta p_3^2], \tag{3.5}$$

where $I(x, y) = b_1 x^2 + (b_2 - a_1)xy - a_2 y^2$.

Let $l_+ = \{(\bar{p}_1 \lambda, \bar{p}_2 \lambda, 1) | \lambda \in \mathbb{R}\}$ be a straight line on Π_3^+, where $\bar{p}_1 = p_1 / \sqrt{1 - p_3^2}$, $\bar{p}_2 = p_2 / \sqrt{1 - p_3^2}$. By direct computation we obtain that on l_+, writing $\mathbf{x} = (\bar{p}_1 \lambda, \bar{p}_2 \lambda, 1)$, we have

$$f(\lambda) := \bar{p}_1 (Q_2(\bar{\mathbf{x}}) - 2\bar{x}_2 Q_3(\bar{\mathbf{x}})) - \bar{p}_2 (Q_1(\bar{\mathbf{x}}) - 2\bar{x}_1 Q_3(\bar{\mathbf{x}})) = \lambda I(\bar{p}_1, \bar{p}_2) + \bar{p}_1 \eta, \qquad \eta = 0, 1.$$

If $\eta = 0$, then it follows from (3.5) that $I(\bar{p}_1, \bar{p}_2) = I(p_1, p_2)/(1 - p_3^2) = 0$. This means that $f(\lambda) \equiv 0$. Therefore, l_+ is an invariant straight line of W_3^+. Let S be the great circle containing the points $(\bar{p}_1, \bar{p}_2, 0)$ and $(0, 0, \pm 1)$. Clearly, $\mathbf{p} \in S$. By Proposition 2.2, the half-great circle $S \cap H_3^+$ is an integral curve of the vector field $\mathbf{Q}_T$. Thus $\mathbf{p}$ can no be a center of $\mathbf{Q}_T$. □

Next consider the critical point $(0, 0, \pm 1)$ of $\mathbf{Q}_T$ with $\eta = 0$. We need the following result.

Lemma 3.4 ([8]). *The origin is a center of the following system*

$$\frac{dx}{dt} = ax + by + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$
$$\frac{dy}{dt} = cx - ay + b_{20}x^2 + b_{11}xy + b_{02}y^2,$$

with $a^2 + bc < 0$, if and only if one of the following conditions holds:

(1) $A\alpha - B\beta = \gamma = 0$,

(2) $\alpha = \beta = 0$,

(3) $5A - \beta = 5B - \alpha = \delta = 0$,

where $A = a_{20} + a_{02}$, $B = b_{20} + b_{02}$, $\alpha = a_{11} + 2b_{02}$, $\beta = b_{11} + 2a_{20}$, $\gamma = b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3$, and $\delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B$.

Proposition 3.5. *Assume that $\eta = 0$. The points $(0, 0, \pm 1)$ are centers of $\mathbf{Q}_T$ on $\mathbb{S}^2$ if and only if*

$$a_1 + b_2 = 4c_0, \qquad (b_2 - a_1)^2 + 4a_2b_1 < 0. \tag{3.6}$$

Proof. To study the singularity $(0, 0, 1)$, we will use the induced system on Π_3^+, which is (3.4) with $\eta = 0$.

Clearly, $\phi_+^3(0, 0, 1) = (0, 0, 1)$. The characteristic equation of the linear approximation system of (3.4) at the singularity $(0, 0)$ is

$$\lambda^2 - (a_1 + b_2 - 4c_0)\lambda + (a_1 - 2c_0)(b_2 - 2c_0) - a_2b_1 = 0.$$

The singularity $(0, 0)$ is a center or focus of the system (3.4) if and only if

$$a_1 + b_2 - 4c_0 = 0, \qquad (a_1 - 2c_0)(b_2 - 2c_0) - a_2b_1 > 0,$$

which is equivalent to (3.6). By straightforward calculations, we find that $A\alpha - B\beta = \gamma = 0$, where $A = -2c_1, B = -2c_2$, $\alpha = -6c_2, \beta = -6c_1$. Therefore, $(0, 0)$ is a center of system (3.4) if and only if the relation (3.6) is satisfied.

By applying the Proposition 2.1, we can conclude that the relations (3.6) are also the sufficient and necessary conditions for $(0, 0, -1)$ to be a center of $\mathbf{Q}_T$ on $\mathbb{S}^2$. □

Let us consider the case $\eta = 1$.

Proposition 3.6. *Assume that $\eta = 1$, then $\mathbf{Q}_T$ has at least a center on $\mathbb{S}^2 \cap H_3^+$ if and only if one of the conditions of Theorem 3.1 (B) is satisfied. Moreover, if the condition (i) $(i \in \{1, 2, 3\})$ of Theorem 3.1 (B) holds, then $\mathbf{Q}_T$ has a unique center $\mathbf{y}_i$ on $\mathbb{S}^2 \cap H_3^+$ and has a unique center $-\mathbf{y}_i$ on $\mathbb{S}^2 \cap H_3^-$ which satisfy $\phi_+^3(\mathbf{y}_i) = (x_i^*, y_i^*, 1)$, where x_i^* and y_i^* are defined in (3.1).*

Proof. Suppose that $\mathbf{p} \in H_3^+ \cap \mathbb{S}^2$ is a singularity of $\mathbf{Q}_T$, and let $(x_0, y_0, 1) = \phi_3^+(\mathbf{p})$. It is easy to see that (x_0, y_0) is a singularity of system (3.4). By taking the transformation $u = \bar{x}_1 - x_0, v = \bar{x}_2 - y_0$, we change system (3.4) to

$$\begin{aligned} \frac{du}{d\bar{\tau}} &= (\bar{a}_1 - 4c_1x_0 - 2c_2y_0)u + (a_2 - 2c_2x_0)v - 2c_1u^2 - 2c_2uv, \\ \frac{dv}{d\bar{\tau}} &= (b_1 - 2c_1y_0)u + (\bar{b}_2 - 2c_1x_0 - 4c_2y_0)v - 2c_1uv - 2c_2v^2. \end{aligned} \tag{3.7}$$

In what follows we will consider the singularity $(0, 0)$ of system (3.7).

One can check directly that system (3.7) satisfies the condition (1) of Lemma 3.4. Thus the point $(0, 0)$ is a center of system (3.7) if and only if the following two equalities hold

$$\bar{a}_1 - 4c_1x_0 - 2c_2y_0 + \bar{b}_2 - 2c_1x_0 - 4c_2y_0 = 0, \tag{3.8}$$

$$\Delta = (\bar{a}_1 - 4c_1x_0 - 2c_2y_0)^2 + (a_2 - 2c_2x_0)(b_1 - 2c_1y_0) < 0, \tag{3.9}$$

where x_0 and y_0 are the isolated solutions of the following equations

$$\bar{a}_1x_0 + a_2y_0 - 2c_1x_0^2 - 2c_2x_0y_0 = 0, \tag{3.10}$$

$$1 + b_1x_0 + \bar{b}_2y_0 - 2c_1x_0y_0 - 2c_2y_0^2 = 0. \tag{3.11}$$

By equations (3.8) and (3.10), we get $3a_2y_0 = (\bar{b}_2 - 2\bar{a}_1)x_0$. We will now split our discussion into two cases.

Case 1. $a_2 = 0$. By the inequality (3.9) we know that $x_0 \neq 0$. It follows from $3a_2y_0 = (\bar{b}_2 - 2\bar{a}_1)x_0$ that $\bar{b}_2 = 2\bar{a}_1$. Hence under the condition $a_2 = 0$, (3.8)–(3.11) are equivalent to

$$\bar{b}_2 = 2\bar{a}_1,\ 2c_1x_0 + 2c_2y_0 - \bar{a}_1 = 0, \quad b_1x_0 + \bar{a}_1y_0 + 1 = 0, \quad \Delta = (2\bar{a}_1c_1 - 2b_1c_2)x_0 < 0. \tag{3.12}$$

In view of $2\bar{a}_1c_1 - 2b_1c_2 \neq 0$, we can get the solution (x_0, y_0) and then find that (3.12) is equivalent to

$$\begin{gathered} \bar{b}_2 = 2\bar{a}_1,\ \bar{a}_1c_1 - b_1c_2 \neq 0, \qquad \bar{a}_1^2 + 2c_2 < 0, \\ x_0 = \frac{\bar{a}_1^2 + 2c_2}{2\bar{a}_1c_1 - 2b_1c_2} =: x_1^*, \qquad y_0 = -\frac{\bar{a}_1b_1 + 2c_1}{2\bar{a}_1c_1 - 2b_1c_2} =: y_1^*. \end{gathered}$$

Consequently, under the condition $a_2 = 0$, the origin of system (3.7) is a center if and only if

$$\bar{b}_2 = 2\bar{a}_1, \qquad \bar{a}_1c_1 - b_1c_2 \neq 0, \qquad \bar{a}_1^2 + 2c_2 < 0. \tag{3.13}$$

And if the above conditions are satisfied, then $\mathbf{Q}_T$ has a unique center $\mathbf{y}_1 \in \mathbb{S}^2 \cap H_3^+$ such that $\phi_+^3(\mathbf{y}_1) = (x_1^*, y_1^*, 1)$. By the symmetry, we know that $\mathbf{Q}_T$ also has a unique center $-\mathbf{y}_1 \in \mathbb{S}^2 \cap H_3^-$ if the condition (3.13) is satisfied.

Case 2. $a_2 \neq 0$. By $3a_2y_0 = (\bar{b}_2 - 2\bar{a}_1)x_0$ and (3.8) we obtain

$$a_2(\bar{a}_1 + \bar{b}_2) = (6a_2c_1 + 2\bar{b}_2c_2 - 4\bar{a}_1c_2)x_0.$$

If $6a_2c_1 + 2\bar{b}_2c_2 - 4\bar{a}_1c_2 = 0$, then $\bar{a}_1 = -\bar{b}_2$ and thus (3.8)–(3.11) are equivalent to

$$c_1x_0 + c_2y_0 = 0, \quad \bar{a}_1x_0 + a_2y_0 = 0, \quad 1 + b_1x_0 - \bar{a}_1y_0 = 0, \quad \Delta = \bar{a}_1^2 + a_2b_1 - 2b_1c_2x_0 < 0. \tag{3.14}$$

Further, by using (3.14), (3.8)–(3.11) are equivalent to

$$\bar{a}_1 = -\bar{b}_2,\ c_1 a_2 = \bar{a}_1 c_2, a_2 b_1 + \bar{a}_1^2 \neq 0, \qquad \Delta = \frac{\bar{a}_1^4 + 2\bar{a}_1^2 a_2 b_1 + a_2^2 b_1^2 + 2a_2 b_1 c_2}{a_2 b_1 + \bar{a}_1^2} < 0,$$
$$x_0 = -\frac{a_2}{a_2 b_1 + \bar{a}_1^2} =: x_2^*, \qquad y_0 = \frac{\bar{a}_1}{a_2 b_1 + \bar{a}_1^2} =: y_2^*.$$

It turns out that, under the condition $a_2 \neq 0$ and $6a_2 c_1 + 2\bar{b}_2 c_2 - 4\bar{a}_1 c_2 = 0$, the origin of system (3.7) is a center if and only if

$$\bar{a}_1 = -\bar{b}_2, \qquad c_1 a_2 = \bar{a}_1 c_2, a_2 b_1 + \bar{a}_1^2 \neq 0, \qquad \frac{\bar{a}_1^4 + 2\bar{a}_1^2 a_2 b_1 + a_2^2 b_1^2 + 2a_2 b_1 c_2}{a_2 b_1 + \bar{a}_1^2} < 0. \tag{3.15}$$

If (3.15) is true, then $\mathbf{Q}_T$ has a unique center $\mathbf{y}_2 \in \mathbb{S}^2 \cap H_3^+$ such that $\phi_+^3(\mathbf{y}_2) = (x_2^*, y_2^*, 1)$. By the symmetry, we know that $\mathbf{Q}_T$ also has a unique center $-\mathbf{y}_2 \in \mathbb{S}^2 \cap H_3^-$ if the condition (3.15) is satisfied.

Next assume that

$$6a_2 c_1 + 2\bar{b}_2 c_2 - 4\bar{a}_1 c_2 \neq 0.$$

The equations (3.8) and (3.10) have a unique solution

$$x_0 = \frac{a_2(\bar{a}_1 + \bar{b}_2)}{6a_2 c_1 + 2\bar{b}_2 c_2 - 4\bar{a}_1 c_2} =: x_3^*, \qquad y_0 = \frac{(\bar{b}_2 - 2\bar{a}_1)(\bar{b}_2 + \bar{a}_1)}{6(3a_2 c_1 + \bar{b}_2 c_2 - 2\bar{a}_1 c_2)} =: y_3^*.$$

Substituting into (3.11) yields

$$2\bar{a}_1^3 - 3\bar{a}_1^2 \bar{b}_2 + 9\bar{a}_1 a_2 b_1 - 3\bar{a}_1 \bar{b}_2^2 - 36\bar{a}_1 c_2 + 9a_2 b_1 \bar{b}_2 + 2\bar{b}_2^3 + 54a_2 c_1 + 18\bar{b}_2 c_2 = 0.$$

Moreover, it follows that

$$\Delta = \frac{1}{9(3a_2 c_1 + \bar{b}_2 c_2 - 2\bar{a}_1 c_2)} \cdot (9\bar{a}_1^2 a_2 c_1 + 27a_2^2 b_1 c_1 - 9\bar{a}_1 a_2 \bar{b}_2 c_1 + 9a_2 \bar{b}_2^2 c_1 - 8\bar{a}_1^3 c_2$$
$$- 27\bar{a}_1 a_2 b_1 c_2 + 12\bar{a}_1^2 \bar{b}_2 c_2 - 6\bar{a}_1 \bar{b}_2^2 c_2 + \bar{b}_2^3 c_2) < 0.$$

This complete the proof. □

Next we are going to study the singularities of $\mathbf{Q}_T$ on the equator.

Proposition 3.7. *(i) If $c_1 = c_2 = 0$, then the equator is a singular circle of $\mathbf{Q}_T$. In other words, each point of the equator is a singularity of $\mathbf{Q}_T$.*

(ii) If $c_1^2 + c_2^2 \neq 0$, then on the equator $\mathbf{Q}_T$ has two singularities at $E_1 = E$ and $E_2 = -E$ respectively, where $E\left(c_2/\sqrt{c_1^2+c_2^2}, -c_1/\sqrt{c_1^2+c_2^2}, 0\right)$. The direction of $\mathbf{Q}_T$ on the equator is shown in Figure 3.1.

Proof. The conclusion follows directly from

$$\mathbf{Q}_T|_{(\pm p_1, \pm p_2, 0)} = (0, 0, \pm 2c_1 p_1 \pm 2c_2 p_2).$$ □

Proposition 3.8. *$\mathbf{Q}_T$ has centers on the equator if and only if $J(c_1, c_2) \neq 0$. Moreover, if $J(c_1, c_2) > 0$ (resp. $J(c_1, c_2) < 0$), then $\mathbf{Q}_T$ has a unique center at E_1 (resp. E_2).*

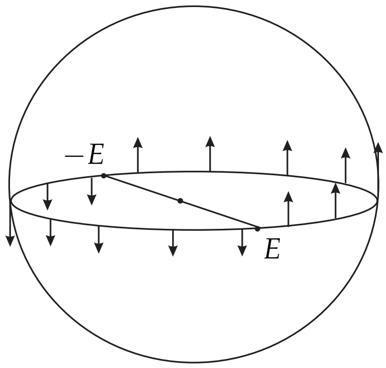

Figure 3.1: The direction of $\mathbf{Q}_T$ on the equator.

Proof. By Proposition 3.7, the equator contains centers only if $c_1^2 + c_2^2 \neq 0$. In what follows we will split our discussion into three cases.

Case 1. $c_2 > 0$. Obviously, $E_1 \in H_1^+$, $E_2 \in H_1^-$. By taking the transformation $\phi_+^1 : H_1^+ \cap \mathbb{S}^2 \to \Pi_1^+$, we obtain the induced system of $\mathbf{Q}_T$ on Π_1^+:

$$\begin{aligned} \frac{d\bar{x}_2}{d\bar{\tau}} &= \bar{x}_3(b_1 + (b_2 - a_1)\bar{x}_2 - a_2\bar{x}_2^2 + \eta x_3^2) =: p_1^+(\bar{x}_2, \bar{x}_3), \\ \frac{d\bar{x}_3}{d\bar{\tau}} &= c_1 + c_2\bar{x}_2 + \left(c_0 - \frac{1}{2}a_1\right)\bar{x}_3^2 - \frac{1}{2}a_2\bar{x}_2\bar{x}_3^2 =: q_1^+(\bar{x}_2, \bar{x}_3), \end{aligned} \tag{3.16}$$

where $\eta = 0, 1$ and $d\bar{\tau} = (2y_1^2 + 2y_2^2 + y_3^2)\sqrt{y_1}d\tau$. In particular, $\phi_+^1(E_2) = (1, -c_1/c_2, 0)$. The characteristic equation of the linear approximation system of (3.16) at the critical point $(-c_1/c_2, 0)$ is $\lambda^2 + J(c_1, c_2)/c_2 = 0$.

Therefore, if $J(c_1, c_2) < 0$, then $(-c_1/c_2, 0)$ is a saddle of system (3.16). If $J(c_1, c_2) > 0$, then $(-c_1/c_2, 0)$ is a center or a focus. Noting that

$$(p_1^+(\bar{x}_2, -\bar{x}_3), q_1^+(\bar{x}_2, -\bar{x}_3)) = (-p_1^+(\bar{x}_2, \bar{x}_3), q_1^+(\bar{x}_2, \bar{x}_3)),$$

we conclude that the critical point $(-c_1/c_2, 0)$ is a center. If $J(c_1, c_2) = 0$, then the equation (3.16) becomes

$$\begin{aligned} \frac{d\bar{x}_2}{d\bar{\tau}} &= -\bar{x}_3\left(\bar{x}_2 + \frac{c_1}{c_2}\right)\left(a_2\bar{x}_2 - \frac{b_1c_2}{c_1}\right) + \eta\bar{x}_3^3, \\ \frac{d\bar{x}_3}{d\bar{\tau}} &= \frac{1}{c_2}\left[2c_2^2\left(\bar{x}_2 + \frac{c_1}{c_2}\right) + (a_2c_1 - a_1c_2 + 2c_0c_2)\bar{x}_3^2 - a_2c_2\left(\bar{x}_2 + \frac{c_1}{c_2}\right)\bar{x}_3^2\right]. \end{aligned} \tag{3.17}$$

In the case $\eta = 0$, the straight line $\bar{x}_2 = -c_1/c_2$ is an invariant line of system (3.16) when $a_2c_1 - a_1c_2 + 2c_0c_2 \neq 0$ and it is a singular line when $a_2c_1 - a_1c_2 + 2c_0c_2$. In any case, the critical point $(-c_1/c_2, 0)$ cannot be a center.

When $\eta = 1$, on the straight line $\bar{x}_2 = -c_1/c_2$ we have $d\bar{x}_2/d\bar{\tau} = \bar{x}_3^3$, meaning that the orbits of system (3.17) pass through the straight line $\bar{x}_2 = -c_1/c_2$ from the left to the right on the upper half-plane and from the right to the left on the lower half-plane. On the other hand, noting also that the direction of vector field (3.17) at the $\bar{x}_2$ axis is upward on the right-hand side of the critical point $(-c_1/c_2, 0)$ and is downward on the left-hand side of the critical point $(-c_1/c_2, 0)$. See Figure 3.2. We conclude that there is no closed orbit around the singularity $(-c_1/c_2, 0)$. So the singularity $(-c_1/c_2, 0)$ is not a center.

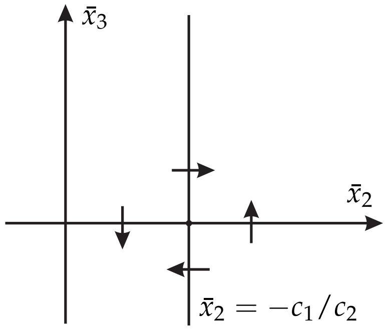

Figure 3.2: The direction of system (3.17).

Next we study the singular point E_2 of $\mathbf{Q}_T$. With the transformation $\phi^1_- : H_1^- \cap \mathbb{S}^2 \to \Pi_1^-$, we obtain the induced system of $\mathbf{Q}_T$ on Π_1^-:

$$\begin{aligned} \frac{d\bar{x}_2}{d\bar{\tau}} &= \bar{x}_3(-b_1 + (b_2 - a_1)\bar{x}_2 + a_2\bar{x}_2^2 + \eta\bar{x}_3^2) =: p_1^-(\bar{x}_2, \bar{x}_3), \\ \frac{d\bar{x}_3}{d\bar{\tau}} &= -c_1 + c_2\bar{x}_2 + (c_0 - \frac{1}{2}a_1)\bar{x}_3^2 + \frac{1}{2}a_2\bar{x}_2\bar{x}_3^2 =: q_1^-(\bar{x}_2, \bar{x}_3), \end{aligned} \tag{3.18}$$

where $\eta = 0, 1$ and $d\bar{\tau} = (2y_1^+ 2y_2^2 + y_3^2)\sqrt{y_1}\, d\tau$. Moreover, $\phi^1_-(E_2) = (1, c_1/c_2, 0)$. The characteristic equation of the linear approximation system of (3.18) at the critical point $(c_1/c_2, 0)$ is

$$\lambda^2 - \frac{1}{c_2}J(c_1, c_2) = 0, \quad \text{where} \quad J(x, y) = a_2x^2 + (b_2 - a_1)xy - b_1y^2. \tag{3.19}$$

In the same way, we can easily verify that the critical point $(c_1/c_2, 0)$ is a center if and only if $J(c_1, c_2) < 0$.

Case 2. $c_2 < 0$. Obviously, $E_1 \in H_1^-$ and $E_2 \in H_1^+$. We make the transformation $\phi^1_- : H_1^- \cap \mathbb{S}^2 \to \Pi_1^-$ and then we obtain the induced system of $\mathbf{Q}_T$ on Π_1^-, which is system (3.18).

By direct calculation we get $\phi^1_-(E_1) = (1, c_1/c_2, 0)$. The characteristic equation of the linear approximation system of (3.18) at the critical point $(c_1/c_2, 0)$ is the equation (3.19). Thus the critical point $(c_1/c_2, 0)$ is a center if and only if $c_2J(c_1, c_2) < 0$.

Using analogous arguments we conclude that E_2 is a center of $\mathbf{Q}_T$ on $\mathbb{S}^2$ if and only if $c_2J(c_1, c_2) > 0$.

Case 3. $c_2 = 0, c_1 \neq 0$. That is, $E_1 = (0, -\operatorname{sgn}(c_1), 0)$, $E_2 = (0, \operatorname{sgn}(c_1), 0)$. Suppose firstly that $c_1 > 0$. By taking the transformation $\phi^2_- : H_2^- \cap \mathbb{S}^2 \to \Pi_2^-$, we obtain the induced system of $\mathbf{Q}_T$ on Π_2^-:

$$\begin{aligned} \frac{d\bar{x}_1}{d\bar{\tau}} &= \bar{x}_3(-a_2 + (a_1 - b_2)\bar{x}_1 + b_1\bar{x}_1^2 + \eta\bar{x}_1\bar{x}_3^2), \\ \frac{d\bar{x}_3}{d\bar{\tau}} &= c_1\bar{x}_1 + (c_0 - \frac{b_2}{2})\bar{x}_3^2 + \frac{b_1}{2}\bar{x}_1\bar{x}_3^2 + \frac{\eta}{2}\bar{x}_3^4. \end{aligned} \tag{3.20}$$

Obviously, $\phi^2_-(E_1) = (0, 0)$.

The characteristic equation of the linear approximation system of (3.20) at the critical point $(0, 0)$ is $\lambda^2 + a_2c_1 = 0$. By similar arguments as above, we find that the critical point $(0, 0)$ is a center of system (3.20) if and only if $a_2c_1 > 0$, i.e., $c_1J(c_1, 0) > 0$.

Analogously, the singularity $E_2 = (0, 1, 0)$ is a center of $\mathbf{Q}_T$ on $\mathbb{S}^2$ if and only if $c_1J(c_1, 0) < 0$.

The case that $c_1 < 0$ can be studied in a similar way, we omit the discussion for the sake of brevity.

In conclusion, the vector field $\mathbf{Q}_T$ has a center on the equator if and only if $J(c_1, c_2) \neq 0$. $\square$

Remark 3.9. If $\eta = 0$ and $a_1 + b_2 = 4c_0$, $(b_2 - a_1)^2 + 4a_2b_1 < 0$, then $J(c_1, c_2) = a_2c_1^2 + (b_2 - a_1)c_1c_2 - b_1c_2^2$ and it is different from zero except for $c_1 = c_2 = 0$. It follows from Proposition 3.5 that $\mathbf{Q}_T$ has exactly three centers at $(0,0,1)$, $(0,0,-1)$, and E or $-E$.

Proof of Theorem 3.1. The proof follows from Propositions 3.3, 3.5, 3.6, 3.8 and Remark 3.9. $\square$

3.2 Quasi-homogeneous systems with weight $(m, m, 1)$

Throughout this subsection we suppose that $\mathbf{Q}_T$ is the projective vector field of system (2.14).

Proof of Theorem 3.2. It is easy to see that the equator of $\mathbb{S}^2$ is a singular great circle of $\mathbf{Q}_T$. Thus, the center (if exists) is located on $\mathbb{S}^2 \cap H_3^{\pm}$.

A direct computation shows that, the induced systems of $\mathbf{Q}_T$ on Π_3^+ and Π_3^- are respectively

$$\frac{d\bar{x}_1}{d\bar{\tau}} = (a_1 - m)\bar{x}_1 + a_2\bar{x}_2, \qquad \frac{d\bar{x}_2}{d\bar{\tau}} = \eta + b_1\bar{x}_1 + (b_2 - m)\bar{x}_2 \tag{3.21}$$

and

$$\frac{d\bar{x}_1}{d\bar{\tau}} = (m - a_1)\bar{x}_1 - a_2\bar{x}_2, \qquad \frac{d\bar{x}_2}{d\bar{\tau}} = (-1)^{m+1}\eta - b_1\bar{x}_1 + (m - b_2)\bar{x}_2, \tag{3.22}$$

where $\eta = 0, 1$ and $d\bar{\tau} = (my_1^2 + my_2^2 + y_3^2)|y_3|d\tau$.

System (3.21) (resp. (3.22)) has a center if and only if $a_1 + b_2 = 2m$, $(a_1 - m)(b_2 - m) - a_2b_1 > 0$, which is equivalent to (3.2). If (3.2) is true, then the center is $(\eta a_2/\Delta_1, \eta(m - a_1)/\Delta_1)$ (resp. $((-1)^m\eta a_2/\Delta_1, (-1)^m\eta(m - a_1)/\Delta_1)$).

Let $\mathbf{y}^+ = (y_1^+, y_2^+, y_3^+) \in \mathbb{S}^2$ such that

$$\left(\frac{\eta a_2}{\Delta_1}, \frac{\eta(m - a_1)}{\Delta_1}, 1\right) = \phi_+^3(\mathbf{y}^+) = \left(\frac{y_1^+}{(y_3^+)^m}, \frac{y_2^+}{(y_3^+)^m}, 1\right), \qquad y_3^+ > 0.$$

Then

$$(y_1^+, y_2^+, y_3^+) = \left(\frac{a_2\lambda_0^m}{\Delta_1}, \frac{(m - a_1)\lambda_0^m}{\Delta_1}, \lambda_0\right), \qquad \eta = 1,$$

or $(y_1^+, y_2^+, y_3^+) = (0, 0, 1)$, $\eta = 0$. Similarly, let $\mathbf{y}^- = (y_1^-, y_2^-, y_3^-) \in \mathbb{S}^2$ such that

$$\left(\frac{(-1)^m\eta a_2}{\Delta_1}, \frac{(-1)^m\eta(m - a_1)}{\Delta_1}, -1\right)) = \phi_-^3(\mathbf{y}^-) = \left(\frac{y_1^-}{(y_3^-)^m}, \frac{y_2^-}{(y_3^-)^m}, -1\right), \qquad y_3^- < 0.$$

Then

$$(y_1^-, y_2^-, y_3^-) = \left(\frac{(-1)^m a_2\lambda_0^m}{\Delta_1}, \frac{(-1)^m(m - a_1)\lambda_0^m}{\Delta}, -\lambda_0\right), \qquad \eta = 1$$

or $(y_1^-, y_2^-, y_3^-) = (0, 0, -1)$, $\eta = 0$, where λ_0 is the unique positive solution of (3.3). $\square$

4 Center of the quasi-homogeneous systems with weight $(2, 2, 3)$ and $(3, 3, 2)$

This section is devoted to derive the sufficient and necessary conditions for the projective vector field $\mathbf{Q}_T$ of systems (2.15) and (2.16) to possess at least one center. The main results of this section are the following two theorems.

Theorem 4.1. *Suppose that $\mathbf{Q}_T$ is the projective vector field of system (2.15), then $\mathbf{Q}_T$ has at least one center on $\mathbb{S}^2$ if and only if one of the following conditions holds:*

(1) $c_1 \neq 0$, $\Delta_2 = c_2^2 - 4c_1c_3 > 0$, $2c_1 + c_2 + \sqrt{\Delta_2} \neq 0$,

(2) $c_1 = 0$, $c_2 < 0$ *or* $c_1 = 0$, $c_2 > 0$, $c_2 + c_3 > 0$.

Moreover,

(i) If (1) *is satisfied and* $2c_1 + c_2 + \sqrt{\Delta_2} > 0$ *(resp.* < 0*), then* $\mathbf{Q}_T$ *has exactly two centers at* $\pm G_1$ *(resp.* $\pm G_2$*), where*

$$G_i = \left(\frac{-c_2 + (-1)^i\sqrt{\Delta_2}}{\sqrt{4c_1^2 + ((-1)^i c_2 - \sqrt{\Delta_2})^2}}, \frac{2c_1}{\sqrt{4c_1^2 + ((-1)^i c_2 - \sqrt{\Delta_2})^2}}, 0 \right), \qquad i = 1, 2. \tag{4.1}$$

(ii) Suppose that (2) *is satisfied. If* $c_2 < 0, c_2 + c_3 > 0$*, then* $\mathbf{Q}_T$ *has exactly four centers at* $\pm(0,0,1)$ *and* $\pm D$*; If* $c_2 < 0, c_2 + c_3 \leq 0$*, then* $\mathbf{Q}_T$ *has exactly two centers at* $\pm(0,0,1)$*; if* $c_2 + c_3 > 0$*,* $c_2 > 0$*, then* $\mathbf{Q}_T$ *has exactly two centers at* $\pm D$*. Here*

$$D = \left(\frac{-c_3}{\sqrt{c_2^2 + c_3^2}}, \frac{c_2}{\sqrt{c_2^2 + c_3^2}}, 0 \right).$$

Theorem 4.2. *The projective vector field* $\mathbf{Q}_T$ *of system (2.16) has no centers on* $\mathbb{S}^2$*.*

To prove our results, we need the following lemma, which is a part of the Nilpotent Singular Points Theorem. The readers are referred to [3] for the complete result.

Lemma 4.3. *Let* $(0,0)$ *be the isolate singularity of system*

$$\frac{dx}{dt} = y + A(x,y), \qquad \frac{dy}{dt} = B(x,y), \tag{4.2}$$

where A *and* B *are analytic in a neighborhood of* $(0,0)$ *and also* $j_1A(0,0) = j_1B(0,0) = 0$*. Let* $y = f(x)$ *be the solution* $y + A(x,y) = 0$ *in a neighborhood of the point* $(0,0)$*. And let* $F(x) = B(x, f(x))$*,* $G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$*. If* $F(x) = ax^m + 0(x^m)$ *and* $G(x) = bx^n + 0(x^n)$ *with* $m, n \in \mathbb{N}, m \geq 2, n \geq 1$ *and* $ab \neq 0$*, then we have*

(i) if m *is even and* $m < 2n + 1$*, then the origin of system (4.2) is a cusp.*

(ii) if m *is even and* $m > 2n + 1$*, then the origin of system (4.2) is a saddle-node.*

Let us consider firstly system (2.15).

Proof of Theorem 4.1. The induced systems of $\mathbf{Q}_T$ on Π_3^+ is

$$\frac{d\bar{x}_1}{d\bar{\tau}} = 1 - \frac{2}{3}\bar{x}_1(c_1\bar{x}_1^2 + c_2\bar{x}_1\bar{x}_2 + c_3\bar{x}_2^2), \qquad \frac{d\bar{x}_2}{d\bar{\tau}} = 1 - \frac{2}{3}\bar{x}_2(c_1\bar{x}_1^2 + c_2\bar{x}_1\bar{x}_2 + c_3\bar{x}_2^2). \tag{4.3}$$

System (4.3) has an isolated singularity if and only if $c_1 + c_2 + c_3 \neq 0$. Assume that $c_1 + c_2 + c_3 \neq 0$, then system (4.3) has a unique isolated singularity at the point

$$E\left(3^{1/3}(2c_1 + 2c_2 + 2c_3)^{-1/3}, 3^{1/3}(2c_1 + 2c_2 + 2c_3)^{-1/3}\right).$$

The characteristic equation of the linear approximation system of (4.3) at the singularity E is

$$3^{1/3}\lambda^2 + 2^{7/3}(c_1+c_2+c_3)^{1/3}\lambda + 6^{2/3}(c_1+c_2+c_3)^{2/3} = 0.$$

Hence it is easy to see that E is not a center of system (4.3). This mean that $\mathbf{Q}_T$ has no centers on $\mathbb{S}^2 \cap H_3^+$. By the symmetry of system (2.15), we know that $\mathbf{Q}_T$ has also no centers on $\mathbb{S}^2 \cap H_3^-$.

By direct computation, we have

$$\mathbf{Q}_T|_{(y_1,y_2,0)} = 2(0,0,c_1y_1^2+c_2y_1y_2+c_3y_2^2), \quad \text{with } y_1^2+y_2^2=1.$$

Therefore,

(i) if $c_2^2-4c_1c_3<0$, then $\mathbf{Q}_T$ has no singularities on the equator; (ii) if $c_2^2-4c_1c_3=0$ and $c_1^2+c_2^2\neq 0$, then $\mathbf{Q}_T$ has exactly two singularities at $G_0\left(\frac{-c_2}{\sqrt{4c_1^2+c_2^2}}, \frac{2c_1}{\sqrt{4c_1^2+c_2^2}}, 0\right)$ and $-G_0$, respectively; If $c_1=c_2=0$, then $\mathbf{Q}_T$ has exactly two singularities at $(1,0,0)$ and $(-1,0,0)$, respectively; (iii) if $c_2^2-4c_1c_3>0$, then $\mathbf{Q}_T$ has exactly four singularities $\pm G_1$ and $\pm G_2$ on the equator, where $\pm G_i$ $(i=1,2)$ are defined in (4.1).

Suppose that $c_1>0$. The induced system of $\mathbf{Q}_T$ giving by $\phi_+^2 : H_2^+\cap\mathbb{S}^2 \to \Pi_2^+$ is

$$\frac{d\bar{x}_1}{d\bar{\tau}} = \bar{x}_3 - \bar{x}_1\bar{x}_3, \qquad \frac{d\bar{x}_3}{d\bar{\tau}} = c_3 + c_2\bar{x}_1 + c_1\bar{x}_1^2 - \frac{3}{2}\bar{x}_3^2. \tag{4.4}$$

Let

$$(\bar{x}_0,1,0) = \begin{cases} \phi_+^2(G_i), & i=1,2, \text{ if } c_2^2-4c_1c_3>0, \\ \phi_+^2(G_0), & i=1,2, \text{ if } c_2^2-4c_1c_3=0. \end{cases}$$

Taking the transformation $u=\bar{x}_1-\bar{x}_0$, $v=\bar{x}_3$, system (4.4) changes to

$$\frac{du}{d\bar{\tau}} = (1-\bar{x}_0)v - uv, \qquad \frac{dv}{d\bar{\tau}} = (c_2+2c_1\bar{x}_0)u + c_1u^2 - \frac{3}{2}v^2. \tag{4.5}$$

The singularity $G_i(i=0,1,2)$ is a center of $\mathbf{Q}_T$ if and only if the origin is a center of system (4.5).

If $c_2^2-4c_1c_3=0$, then $c_2+2c_1\bar{x}_0=0$. Obviously, in this case the origin of system (4.5) is not a center if $1-\bar{x}_0=0$. Thus we assume that $1-\bar{x}_0\neq 0$, which means that, by the time rescaling, system (4.5) can be transformed to the same form as (4.2). By the result of Lemma 4.3 we know that the origin of system (4.5) is a cusp.

If $c_2^2-4c_1c_3>0$, then $c_2+2c_1\bar{x}_0\neq 0$. By Lemma 3.4, the origin is a center of system (4.5) if and only if $(c_2+2c_1\bar{x}_0)(1-\bar{x}_0)<0$. Therefore, using the explicit expression of x_0, G_i (and hence $-G_i$) is center of $\mathbf{Q}_T$ if and only if

$$\frac{(-1)^i\sqrt{c_2^2-4c_1c_3}\left(2c_1+c_2-(-1)^i\sqrt{c_2^2-4c_1c_3}\right)}{2c_1} < 0, \qquad i=1,2.$$

In other words, $\pm G_1$ are centers of $\mathbf{Q}_T$ if and only if

$$c_2^2-4c_1c_3>0, \qquad c_1\left(2c_1+c_2+\sqrt{c_2^2-4c_1c_3}\right)>0,$$

and $\pm G_2$ are centers of $\mathbf{Q}_T$ if and only if

$$c_2^2-4c_1c_3>0, \qquad c_1\left(2c_1+c_2-\sqrt{c_2^2-4c_1c_3}\right)<0.$$

Similarly, if $c_1 < 0$, then by the induced system of $\mathbf{Q}_T$ on Π_2^-, we conclude that $\pm G_1$ are centers of $\mathbf{Q}_T$ if and only if

$$c_2^2 - 4c_1c_3 > 0, \qquad c_1\left(2c_1 + c_2 + \sqrt{c_2^2 - 4c_1c_3}\right) < 0,$$

and $\pm G_2$ are centers of $\mathbf{Q}_T$ if and only if

$$c_2^2 - 4c_1c_3 > 0, \qquad c_1\left(2c_1 + c_2 - \sqrt{c_2^2 - 4c_1c_3}\right) > 0.$$

Finally consider the case that $c_1 = 0$. If $c_2 \neq 0$, then $\mathbf{Q}_T$ has two pairs of singularities on the equator: $\pm(1,0,0)$ and $\pm D$. Using the procedure as above, we conclude that $\pm D$ are centers of $\mathbf{Q}_T$ if and only if $c_2 + c_3 > 0$.

To study the singularities $\pm(1,0,0)$, we use the induced system of $\mathbf{Q}_T$ giving by $\phi_+^1 : H_1^+ \cap \mathbb{S}^2 \to \Pi_1^+$, which is

$$\frac{d\bar{x}_2}{d\bar{\tau}} = \bar{x}_3 - \bar{x}_2\bar{x}_3, \qquad \frac{d\bar{x}_3}{d\bar{\tau}} = c_2\bar{x}_2 + c_3\bar{x}_2^2 - \frac{3}{2}\bar{x}_3^2. \tag{4.6}$$

And $\phi_+^1(1,0,0) = (0,0,0)$. If $c_2 \neq 0$, then by the result of Lemma 3.4 we know that the origin is a center of system (4.6) if and only if $c_2 < 0$. If $c_2 = 0$, then $\pm(1,0,0)$ is the unique pair of singularities of $\mathbf{Q}_T$ on the equator. By Lemma 4.3, the origin is not a center of system (4.6), meaning that $\mathbf{Q}_T$ has no centers in this case. □

We are now in the position to prove the result for system (2.16).

Proof of Theorem 4.2. The induced system of $\mathbf{Q}_T$ on Π_3^+ is

$$\frac{d\bar{x}_1}{d\bar{\tau}} = 1 - \frac{3}{2}\bar{x}_1(c_1\bar{x}_1 + c_2\bar{x}_2), \qquad \frac{d\bar{x}_2}{d\bar{\tau}} = 1 - \frac{3}{2}\bar{x}_2(c_1\bar{x}_1 + c_2\bar{x}_2). \tag{4.7}$$

System (4.7) has an isolated singularity if and only if $c_1 + c_2 > 0$. And if $c_1 + c_2 > 0$, then system (4.3) has a unique isolated singularity at

$$\left(\sqrt{\frac{2}{3c_1 + 3c_2}}, \sqrt{\frac{2}{3c_1 + 3c_2}}\right).$$

By direct computation, we obtain that the eigenvalues of the linear approximation of system of (4.7) at that singularity are

$$\lambda_1 = \lambda_2 = \sqrt{\frac{3c_1 + 3c_2}{2}} > 0.$$

Hence it is not a center of system (4.3), meaning that $\mathbf{Q}_T$ has no centers on $\mathbb{S}^2 \cap H_3^+$. By the symmetry of (4.7), we know that $\mathbf{Q}_T$ also has no centers on $\mathbb{S}^2 \cap H_3^-$.

At the equator, we have

$$\mathbf{Q}_T|_{(y_1,y_2,0)} = 3(0,0,c_1y_1 + c_2y_2), \quad \text{with } y_1^2 + y_2^2 = 1.$$

Therefore, $\mathbf{Q}_T$ has two singularities at $E\Big(-c_2/\sqrt{c_1^2 + c_2^2},\, c_1/\sqrt{c_1^2 + c_2^2}, 0\Big)$ and $-E$ respectively.

Assume firstly that $c_1 > 0$. Then $E \in H_2^+ \cap \mathbb{S}^2$. The induced system of $\mathbf{Q}_T$ giving by $\phi_+^2 : H_2^+ \cap \mathbb{S}^2 \to \Pi_2^+$ is

$$\frac{d\bar{x}_1}{d\bar{\tau}} = \bar{x}_3^2 - \bar{x}_1\bar{x}_3^2, \qquad \frac{d\bar{x}_3}{d\bar{\tau}} = c_2 + c_1\bar{x}_1 - \frac{3}{2}\bar{x}_3^2. \tag{4.8}$$

And $\phi_+^2(E) = (-c_2/c_1, 0, 0)$.

Taking the transformation $u = \bar{x}_3, v = c_1\bar{x}_1 + c_2$, system (4.8) changes to

$$\frac{du}{d\bar{\tau}} = v - \frac{2}{3}u^2 =: u + A(u,v), \qquad \frac{dv}{d\bar{\tau}} = (c_1 + c_2)u^2 - u^2 v =: B(u,v). \tag{4.9}$$

The singularity E is a center of $\mathbf{Q}_T$ if and only if the origin is a center of system (4.9).

Let $f(u) = \frac{2}{3}u^2$ and $F(u) = B(u, f(u)) = (c_1 + c_2)u^2 - \frac{2}{3}u^5$, $G(u) = (\partial A/\partial u + \partial B/\partial v)(u, f(u)) = -3u^3$. By the result of Lemma 4.3, we know that the origin of system (4.9) is a cusp. Therefore, E (and hence $-E$) is not a center of $\mathbf{Q}_T$. □

5 Center of the quasi-homogeneous systems with weight $(1, 1, n)$

In this section we will study the centers of projective vector field $\mathbf{Q}_T$ of system (1.1) with weight $(1, 1, n)$ and degree $d = 2$. By Theorem 1.1 of [6], the corresponding $\mathbf{Q}_T$ of system (2.11) has no centers on $\mathbb{S}^2$. Thus in the rest of this section we assume that $\mathbf{Q}_T$ is the projective vector field of system (2.12). The main result of this section is the following theorem.

Theorem 5.1. *The $\mathbf{Q}_T$ of system (2.12) has at least one center on $\mathbb{S}^2$ if and only if one of the following conditions is satisfied:*

(1) $a_3 = 0$, $2B_3 + C_2 \neq 0$, $G(x_1^*) = 0$, $F_1(x_1^*) = 0$, $F_2(x_1^*) = 0$, $(2B_3 - C_2)^2 + 8d_0 < 0$, with $x_1^* = -(B_2 + C_1)/(2B_3 + C_2)$,

(2) $a_3 = 2B_3 + C_2 = B_2 + C_1 = 0$, $G(x_2^*) = 0$, $F_1(x_2^*) = 0$, $2B_3^2 + d_0 < 0$, with $x_2^* = -(3B_2B_3 + d_1)/(6B_3^2 + 3d_0)$,

(3) $a_3 = 0, 2B_3 + C_2 \neq 0, G(x_3^*) = 0, F_1(x_3^*) < 0, F_2(x_3^*) = 0$, with $x_3^* = -(B_2 + C_1)/(2B_3 + C_2)$,

(4) $a_3 = 2B_3 + C_2 = B_2 + C_1 = 0$, and there exists x_4^* such that $G(x_4^*) = 0$, $F_1(x_4^*) < 0$,

(5) $a_3 \neq 0$, and there exists x_5^* such that $D(x_5^*) = F_4(x_5^*) = F_3(x_5^*) = G(x_5^*) = 0, F_1(x_5^*) < 0$,

where $B_2 = b_2 - a_1$, $B_3 = b_3 - a_2$, $C_1 = c_1 - 2a_1$, $C_2 = c_2 - 2a_2$, and

$$\begin{aligned}
G(x) &= d_3 - b_1c_1 + (d_2 - b_1C_2 - B_2C_1)x + (d_1 + 2a_3b_1 - B_3C_1 - B_2C_2)x^2 \\
&\quad + (d_0 + 2a_3B_2 + a_3C_1 - B_3C_2)x^3 + a_3(2B_3 + C_2)x^4 - 2a_3^2x^5, \\
D(x) &= B_2 + C_1 + (2B_3 + C_2)x - 5a_3x^2, \\
F_1(x) &= B_2^2 - b_1C_2 + d_2 + (4B_2B_3 - B_2C_2 + 4a_3b_1 + 2d_1)x \\
&\quad + (4B_3^2 - B_3C_2 - 2a_3B_2 + 3d_0)x^2 - a_3(8B_3 - C_2)x^3 + 5a_3^2x^4, \\
F_2(x) &= -B_2B_3 + B_2C_2 - d_1 + (2B_3C_2 - 2B_3^2 - 3d_0)x, \\
F_3(x) &= 5a_3F_1(x) - (B_3 + C_2 - 7a_3x)F_2(x), \\
F_4(x) &= 13B_3C_2 - 25a_3B_2 - 12B_3^2 - 3C_2^2 - 25d_0 - 15a_3(2B_3 + C_2)x + 75a_3^2x^2.
\end{aligned}$$

Moreover, if the condition (i) $(i \in \{1, 2, 3, 4, 5\})$ holds, then $(\pm y_1^*, \pm y_2^*, y_3^*) \in \mathbb{S}^2$ are two centers of $\mathbf{Q}_T$ if and only if $y_2^* = y_1^*x_i^*$ and $y_3^* = (y_1^*)^2 f(x_i^*)$, where $f(x) = a_3x^3 - B_3x^2 - B_2x - b_1$.

We obtain from Theorem 5.1 the following corollary.

Corollary 5.2. *The set of number of centers of the projective vector field* $\mathbf{Q}_T$ *of system* (2.12) *on* $\mathbb{S}^2$, *for all possible* $\mathbf{Q}$ *with weight* $(1,1,2)$ *and degree* 2, *is* $\{0,2,4\}$.

Condition (5) of Theorem 5.1 can be replaced by the following criterion which is more convenient to be manipulated with a computer.

Corollary 5.3. *The* $\mathbf{Q}_T$ *of system* (2.12) *with* $a_3 \neq 0$ *has at least one center on* $\mathbb{S}^2$ *if and only if one of the following conditions is satisfied:*

1. $\beta_1 \neq 0$, $D(x^*_{5,1}) = 0$, $F_1(x^*_{5,1}) < 0$, $F_3(x^*_{5,1}) = 0$, $F_4(x^*_{5,1}) = 0$, *with* $x^*_{5,1} = \alpha_1/\beta_1$,
2. $\beta_1 = \alpha_1 = 0$, $\Delta_3 \geq 0$, $F_1(x^*_{5,2}) < 0$, $F_3(x^*_{5,2}) = 0$, $F_4(x^*_{5,2}) = 0$, *with* $x^*_{5,2} = (2B_3 + C_2 + \sqrt{\Delta_3})/(10a_3)$ *or* $x^*_{5,2} = (2B_3 + C_2 - \sqrt{\Delta_3})/(10a_3)$,

where $\Delta_3 = (2B_3 + C_2)^2 + 20a_3(B_2 + C_1)$,

$$\begin{aligned}\alpha_1 =\ & 250a_3^2b_1B_2 + 110a_3B_2^2B_3 + 24B_2B_3^3 - 375a_3^2b_1C_1 + 45a_3B_2B_3C_1 \\ & + 24B_3^3C_1 - 65a_3B_3C_1^2 - 70a_3B_2^2C_2 - 14B_2B_3^2C_2 - 40a_3B_2C_1C_2 - 14B_3^2C_1C_2 \\ & + 30a_3C_1^2C_2 - 7B_2B_3C_2^2 - 7B_3C_1C_2^2 + 3B_2C_2^3 + 3C_1C_2^3 + 50B_2B_3d_0 \\ & + 50B_3C_1d_0 + 25B_2C_2d_0 + 25C_1C_2d_0 + 125a_3B_2d_1 + 125a_3C_1d_1 + 625a_3^2d_3,\end{aligned}$$

and

$$\begin{aligned}\beta_1 =\ & -(200a_3^2B_2^2 + 500a_3^2b_1B_3 + 280a_3B_2B_3^2 + 48B_3^4 - 350a_3^2B_2C_1 - 70a_3B_3^2C_1 \\ & + 75a_3^2C_1^2 - 375a_3^2b_1C_2 - 95a_3B_2B_3C_2 - 4B_3^3C_2 - 70a_3B_3C_1C_2 - 55a_3B_2C_2^2 \\ & - 28B_3^2C_2^2 + 45a_3C_1C_2^2 - B_3C_2^3 + 3C_2^4 + 125a_3B_2d_0 + 100B_3^2d_0 + 125a_3C_1d_0 \\ & + 100B_3C_2d_0 + 25C_2^2d_0 + 250a_3B_3d_1 + 125a_3C_2d_1 + 625a_3^2d_2).\end{aligned}$$

Before proving the above results, we give some necessary information about the projective vector field $\mathbf{Q}_T$. Firstly, we have

$$\mathbf{Q}_T|_{y_1=0} = (a_2(1-y_3^4), y_3 - y_2p_4(y_2,y_3), q_4(y_2,y_3)), \quad \text{with } y_2^2 + y_3^2 = 1,$$

where $p_4(y_2,y_3)$ and $q_4(y_2,y_3)$ are two polynomials of degree not more than 4. Therefore, $\mathbf{Q}_T$ has no singularities at $S_1 := \{(y_1,y_2,y_3) \in \mathbb{S}^2 | y_1 = 0\}$ if $a_2 \neq 0$. Suppose that $a_2 = 0$, then the first component of $\mathbf{Q}_T$ is identically zero. This means that S_1 is invariant under the vector field $\mathbf{Q}_T$. In any case, $\mathbf{Q}_T$ has no centers at S_1.

Next let us study the singularity of $\mathbf{Q}_T$ on $\mathbb{S}^2 \setminus S_1$. By the symmetry (see Proposition 2.1), it is enough to study the singularity on $\mathbb{S}^2 \cap H_1^+$. The induced system of $\mathbf{Q}_T$ on Π_1^+ is

$$\begin{aligned}\frac{d\bar{x}_2}{d\bar{\tau}} &= b_1 + B_2\bar{x}_2 + \bar{x}_3 + B_3\bar{x}_2^2 - a_3\bar{x}_2^3, \\ \frac{d\bar{x}_3}{d\bar{\tau}} &= d_3 + d_2\bar{x}_2 + C_1\bar{x}_3 + d_1\bar{x}_2^2 + C_2\bar{x}_2\bar{x}_3 + d_0\bar{x}_2^3 - 2a_3\bar{x}_2^2\bar{x}_3.\end{aligned} \tag{5.1}$$

Suppose that (x_0, y_0) is a singularity of system (5.1), i.e.,

$$\begin{aligned}y_0 = f(x_0) &= a_3x_0^3 - b_1 - B_2x_0 - B_3x_0^2, \\ G(x_0) &= d_3 - b_1c_1 + (d_2 - b_1C_2 - B_2C_1)x_0 + (d_1 + 2a_3b_1 - B_3C_1 - B_2C_2)x_0^2 \\ &\quad + (d_0 + 2a_3B_2 + a_3C_1 - B_3C_2)x_0^3 + a_3(2B_3 + C_2)x_0^4 - 2a_3^2x_0^5 = 0.\end{aligned} \tag{5.2}$$

Taking the transformation $u = \bar{x}_2 - x_0, v = \bar{x}_3 - y_0$, we change system (5.1) to

$$\frac{du}{d\bar{\tau}} = au + v + bu^2 - a_3u^3, \frac{dv}{d\bar{\tau}} = cu + \bar{a}v + du^2 + euv + d_0u^3 - 2a_3u^2v, \tag{5.3}$$

where

$$\begin{aligned} &a = B_2+2B_3x_0-3a_3x_0^2, \quad \bar{a} = C_1+C_2x_0-2a_3x_0^2, \quad b = B_3-3a_3x_0, \\ &c = d_2-b_1C_2+(4a_3b_1-B_2C_2+2d_1)x_0+(4a_3B_2-B_3C_2+3d_0)x_0^2+a_3(4B_3+C_2)x_0^3-4a_3^2x_0^4, \\ &d = d_1+2a_3b_1+(2a_3B_2+3d_0)x_0+2a_3B_3x_0^2-2a_3^2x_0^3, \quad e = C_2-4a_3x_0. \end{aligned} \tag{5.4}$$

Thus, the point (x_0, y_0) is a center of system (5.1) if and only if system (5.3) has a center at the origin.

By regarding x_0 and y_0 as the parameters of system (5.3), we have the following result.

Lemma 5.4. *The origin is a center of system* (5.3) *if and only if* (x_0, y_0) *satisfies one of the following conditions:*

(1) $a_3 \neq 0$, $D(x_0) = 0$ *and* $\alpha_1 - \beta_1 x_0 = 0$;

(2) $a_3 = 0$, $2B_3 + C_2 \neq 0$, $x_0 = -(B_2 + C_1)/(2B_3 + C_2)$ *and* $G(x_0) = 0$;

(3) $a_3 = 2B_3 + C_2 = B_2 + C_1 = 0$, $G_0(x_0) = 0$,

where $G_0 = G|_{a_3=2B_3+C_2=B_2+C_1=0}$, *i.e.,*

$$G_0(x) = b_1B_2 + d_3 + (2b_1B_2 + B_2^2 + d_2)x + (3B_2B_3 + d_1)x^2 + (2B_2B_3 + d_0)x^3.$$

Proof. It is easy to see that system (5.3) has a center at the origin only if (x_0, y_0) satisfies the relations (5.2) and

$$a + \bar{a} = B_2 + C_1 + (2B_3 + C_2)x_0 - 5a_3x_0^2 = D(x_0) = 0. \tag{5.5}$$

If we regard the equality (5.5) as an equation in the variable x_0, then (5.5) has solution if and only if one of the following three conditions is satisfied:

$$\begin{aligned} &(i_1)\ a_3 \neq 0,\ \Delta_3 = (2B_3 + C_2)^2 + 20a_3(B_2 + C_1) \geq 0, \\ &(i_2)\ a_3 = 0,\ 2B_3 + C_2 \neq 0, \\ &(i_3)\ a_3 = 2B_3 + C_2 = B_2 + C_1 = 0. \end{aligned}$$

If $a_3 \neq 0$, then we use (5.5) to reduce the power of x_0 in (5.2) and we obtain that

$$\alpha_1 - \beta_1 x_0 = 0.$$

If $a_3 = 0$, $2B_3 + C_2 \neq 0$, then we get from (5.5) that $x_0 = -(B_2 + C_1)/(2B_3 + C_2)$. Then, substituting into (5.5) yields that $G(-(B_2 + C_1)/(2B_3 + C_2)) = 0$.

Finally, if $a_3 = 2B_3 + C_2 = B_2 + C_1 = 0$, then $a + \bar{a} = 0$ holds automatically and the function G reduces to G_0. □

Remark 5.5. We would like to point out here that if $a_3 = 2B_3 + C_2 = B_2 + C_1 = 0$, then the divergence of system (5.3) is identically zero.

Next, to obtain the sufficient and necessary conditions under which the origin is a center of system (5.3), we will apply a result of [4], which provides a criterion to decide whether the origin is a center of the following Liénard differential equation

$$\dot{x} = y - (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5), \quad \dot{y} = -(b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5). \tag{5.6}$$

Lemma 5.6 ([4])**.** *A system of type* (5.6) *has either a center or a focus at the origin if and only if one of the following three conditions holds:*

(i) $a_1^2 - 4b_1 < 0$,

(ii) $a_1 = b_1 = b_2 = 0$ *and* $2a_2^2 - 4b_3 < 0$,

(iii) $a_1 = a_2 = b_1 = b_2 = b_3 = b_4 = 0$ *and* $3a_3^2 - 4b_5 < 0$.

The next lemma gives the sufficient and necessary conditions under which system (5.6) has a center at the origin. By using the result of the above lemma and for the sake of simplicity we only consider the cases that $b_1 = 1$ or $b_1 = b_2 = 0, b_3 = 1$, or $b_1 = b_2 = b_3 = b_4 = 0, b_5 = 1$.

Lemma 5.7 ([4])**.** *A system of type* (5.6) *has a center at the origin if and only if one of the following conditions holds:*

(i) $b_1 = 1$ *and*

- *(a)* $a_3 = a_5 = b_2 = b_4 = 0$,
- *(b)* $a_2 = a_3 = a_4 = a_5 = 0$,
- *(c)* $a_4 = a_5 = 0, 3a_3 = 2a_2b_2, 3b_5 = 2b_2^2b_3$ *and* $3b_4 = 5b_2b_3$,
- *(d)* $b_5 = 0, 3a_3 = 2a_2b_2, 2a_4 = a_2b_3$ *and* $5a_5 = 2a_2b_4$.

(ii) $b_1 = b_2 = 0,\ b_3 = 1,\ a_2^2 < 2$ *and*

- *(a)* $a_3 = a_5 = b_4 = 0$,
- *(b)* $a_2 = a_3 = a_4 = a_5 = 0$,
- *(c)* $a_2 = a_3 = b_5 = 0, 5a_5 = 4a_4b_4$,
- *(d)* $a_4 = a_5 = 0, 5a_3 = 2a_2b_4$ *and* $25b_5 = 6b_4^2$.

(iii) $b_1 = b_2 = b_3 = b_4 = 0,\ b_5 = 1$ *and* $a_1 = a_2 = a_3 = a_5 = 0$.

Proof of Theorem 5.1. From now on we assume in (5.3) that one of the conditions of Lemma 5.4 is satisfied and hence $\bar{a} = -a$. By taking the transformation $x = u$, $y = v - (\bar{a}u + \frac{1}{2}eu^2 - \frac{2}{3}a_3u^3)$, system (5.3) is changed to the Liénard differential system

$$\frac{dx}{d\bar{\tau}} = y - (\bar{a}_2x^2 + \bar{a}_3x^3), \quad \frac{dy}{d\bar{\tau}} = -(\bar{b}_1x + \bar{b}_2x^2 + \bar{b}_3x^3 + \bar{b}_4x^4 + \bar{b}_5x^5), \tag{5.7}$$

where

$$\bar{a}_2 = -b - \frac{e}{2}, \quad \bar{a}_3 = \frac{5}{3}a_3, \quad \bar{b}_1 = -a^2 - c, \quad \bar{b}_2 = a(e-b) - d,$$
$$\bar{b}_3 = be - a_3a - d_0, \quad \bar{b}_4 = -a_3(2b+e) = 2a_3\bar{a}_2, \quad \bar{b}_5 = 2a_3^2.$$

We apply Lemma 5.7 to prove our results. Noting firstly that $\bar{a}_3 = 0$ implies $\bar{b}_4 = \bar{b}_5 = 0$, we find that the condition (iii) of Lemma 5.7 is not true for system (5.7). Consider the case that $\bar{b}_1 = \bar{b}_2 = 0, \bar{b}_3 \neq 0$. By the second condition of Lemma 5.6 we assume that $\bar{b}_3 > 0$. Taking the transformation $X = \sqrt{\bar{b}_3}x, Y = \sqrt{\bar{b}_3}y$, we change system (5.7) to

$$\frac{dX}{d\tilde{\tau}} = Y - (\tilde{a}_2X^2 + \tilde{a}_3X^3), \qquad \frac{dY}{d\tilde{\tau}} = -(X^3 + \tilde{b}_4X^4 + \tilde{b}_5X^5),$$

where

$$\tilde{a}_2 = \bar{a}_2/(\bar{b}_3)^{1/2}, \qquad \tilde{a}_3 = \bar{a}_3/\bar{b}_3, \qquad \tilde{b}_4 = \bar{b}_4/(\bar{b}_3)^{3/2}, \quad \text{and} \quad \tilde{b}_5 = \bar{b}_5/\bar{b}_3^2.$$

By a careful manipulation we find that the condition (ii) of Lemma 5.7 is equivalent to the following condition (keeping in mind that $\bar{a}_3 = 0$ implies $\bar{b}_4 = \bar{b}_5 = 0$)

$$\begin{aligned} &a^2 + c = a(e-b) - d = 0, \qquad (2b+e)^2 < 8(be - a_3a - d_0) = 8\bar{b}_3, \quad \text{and} \\ &(a)\ a_3 = 0, \quad \text{or} \quad (b)\ 25(be - a_3a - d_0) = 3(2b+e)^2. \end{aligned}$$

Furthermore, as condition (b) can not be satisfied, it follows that the above condition is equivalent to

$$a_3 = 0, \qquad a^2 + c = a(e-b) - d = 0, \quad \text{and} \quad (2b-e)^2 + 8d_0 < 0. \tag{5.8}$$

Using the expressions (5.4), (5.8) turns out to

$$\begin{aligned} &a_3 = 0, \\ &F_1(x_0) = B_2^2 - b_1C_2 + d_2 + (4B_2B_3 - B_2C_2 + 2d_1)x_0 + (4B_3^2 - B_3C_2 + 3d_0)x_0^2 = 0, \\ &F_2(x_0) = -B_2B_3 + B_2C_2 - d_1 + (2B_3C_2 - 2B_3^2 - 3d_0)x_0 = 0, \\ &(2B_3 - C_2)^2 + 8d_0 < 0. \end{aligned} \tag{5.9}$$

Therefore, if $2B_3 + C_2 \neq 0$, then according to Lemma 5.4, we conclude that system (5.3) has a center at the origin if

$$a_3 = 0, \quad 2B_3 + C_2 \neq 0, \quad G(x_1^*) = 0, \quad F_1(x_1^*) = 0, \quad F_2(x_1^*) = 0, \quad (2B_3 - C_2)^2 + 8d_0 < 0,$$

where $x_1^* = -(B_2 + C_1)/(2B_3 + C_2)$ is the unique solution of $D(x) = 0$. This confirms condition (1) of the statement.

If $2B_3 + C_2 = B_2 + C_1 = 0$, then (5.9) reduces to

$$\begin{aligned} &a_3 = 0, \quad F_1(x_0) = B_2^2 + 2b_1B_3 + d_2 + (6B_2B_3 + 2d_1)x_0 + 3(2B_3^2 + d_0)x_0^2 = 0, \\ &F_2(x_0) = -3B_2B_3 - d_1 - 3(2B_3^2 + d_0)x_0 = 0, \quad 2B_3^2 + d_0 < 0. \end{aligned}$$

Thus $F_2(x_0) = 0$ can be replaced by $x_0 = -(3B_2B_3 + d_1)/3(2B_3^2 + d_0) =: x_2^*$. This confirms condition (2) of the statement.

Next consider the case that $\bar{b}_1 \neq 0$. By the first condition of Lemma 5.6 we assume that $\bar{b}_1 > 0$. Taking the transformation $X = \sqrt{\bar{b}_1}x, Y = y, d\tilde{\tau} = \sqrt{\bar{b}_1}d\tau$, we change system (5.7) to

$$\frac{dX}{d\tilde{\tau}} = Y - (\tilde{a}_2X^2 + \tilde{a}_3X^3), \qquad \frac{dY}{d\tilde{\tau}} = -(X + \tilde{b}_2X^2 + \tilde{b}_3X^3 + \tilde{b}_4X^4 + \tilde{b}_5X^5),$$

where

$$\tilde{a}_2 = \bar{a}_2/\bar{b}_1, \quad \tilde{a}_3 = \bar{a}_3/(\bar{b}_1)^{3/2}, \quad \tilde{b}_2 = \bar{b}_2/(\bar{b}_1)^{3/2}, \quad \tilde{b}_3 = \bar{b}_3/\bar{b}_1^2, \quad \tilde{b}_4 = \bar{b}_4/(\bar{b}_1)^{5/2}, \quad \tilde{b}_5 = \bar{b}_5/\bar{b}_1^3.$$

By a careful inspection we find the condition (i) of Lemma 5.7 is equivalent to the following condition (also keeping in mind that $\bar{a}_3 = 0$ implies $\bar{b}_4 = \bar{b}_5 = 0$):

$$\begin{aligned}&a^2+c<0 \text{ and } \quad (a)\ a_3=0,\ a(e-b)-d=0,\ \ (b)\ a_3=0, 2b+e=0,\\&(c)\ 5a_3(a^2+c)=(2b+e)(ae-ab-d),\ 3a_3^2(a^2+c)^2=(ae-ab-d)^2(be-a_3a-d_0),\\&\quad 3a_3(a^2+c)(2b+e)=5(ae-ab-d)(be-a_3a-d_0),\quad a_3\neq 0.\end{aligned}\tag{5.10}$$

Noting further that the three equalities of (c) in (5.10) are equivalent to the more simple equalities

$$(c):\quad 5a_3(a^2+c)=(2b+e)(ae-ab-d),\qquad 25(be-a_3a-d_0)=3(2b+e)^2.$$

Consequently, using (5.4), the condition (5.10) turns out to

$$\begin{aligned}&F_1(x_0)=a^2+c=B_2^2-b_1C_2+d_2+(4B_2B_3-B_2C_2+4a_3b_1+2d_1)x_0\\&\qquad+(4B_3^2-B_3C_2-2a_3B_2+3d_0)x_0^2-a_3(8B_3-C_2)x_0^3+5a_3^2x_0^4<0 \text{ and}\\&(a)\ a_3=0,\ F_2(x_0)=ae-ab-d=-B_2B_3+B_2C_2-d_1+(2B_3C_2-2B_3^2-3d_0)x_0=0,\\&(b)\ a_3=0,\ 2B_3+C_2=0,\\&(c)\ a_3\neq 0,\ F_3(x_0)=5a_3F_1(x_0)-(2b+e)F_2(x_0)=5a_3B_2^2+4a_3b_1B_3+2B_2B_3^2-3a_3b_1C_2\\&\quad-B_2B_3C_2-B_2C_2^2+2B_3d_1+C_2d_1+5a_3d_2+(2B_3+C_2)(8a_3B_2+2B_3^2-2B_3C_2+3d_0)x_0\\&\quad-a_3(40a_3B_2-2B_3^2-22B_3C_2-3C_2^2+15d_0)x_0^2-30a_3^2(2B_3+C_2)x_0^3+75a_3^3x_0^4=0,\\&\quad F_4(x_0)=13B_3C_2-25a_3B_2-12B_3^2-3C_2^2-25d_0-15a_3(2B_3+C_2)x_0+75a_3^2x_0^2=0.\end{aligned}\tag{5.11}$$

A comparison between (a) and (b) means that we can assume in (a) that $2B_3+C_2\neq 0$, and hence condition (3) of Theorem 5.1 follows from condition (a) of (5.11) and Lemma 5.4.

Suppose that $F_1(x_0)<0$ and (b) is true. Then by Lemma 5.4 we must impose another condition $B_2+C_1=0$ on (b). Thus we obtain condition (4) of Theorem 5.1.

Similarly, conditions (5) of the statement of Theorem 5.1 are obtained by using also Lemma 5.4 and the condition (c) of (5.11).

Finally, if the condition (i) $(i\in\{1,2,3,4,5\})$ of Theorem 5.1 is true, then system (5.1) has a center at $(x_i^*, f(x_i^*))$. Hence by the relationship of systems (2.12) and (5.1) as well as the symmetry of $\mathbf{Q}_T$, we know that $(\pm y_1^*, \pm y_2^*, y_3^*)\in\mathbb{S}^2$ are two centers of $\mathbf{Q}_T$ if and only if

$$\phi_1^+(\pm y_1^*, \pm y_2^*, y_3^*)=(1, x_i^*, f(x_i^*)),\quad \text{for } i=1,2,3,4,5,6.\qquad\square$$

Proof of Corollary 5.2. In Theorem 5.1, if one of the conditions (1), (2), (3) is satisfied, then it is clear that $\mathbf{Q}_T$ has exactly two centers on $\mathbb{S}^2$. If condition (5) is satisfied, then by $\deg D(x)\leq 2$ we know that $\mathbf{Q}_T$ has at most four centers on $\mathbb{S}^2$.

Suppose that condition (4) is true. Noting, in this case, that $F_3(x)=G'(x)$. If $G(x)\equiv 0$, then there exists no x_4^* such that $F_3(x_4^*)<0$. So we assume that $G(x)$ is a nonzero polynomial and hence it has at most three zeros. Suppose that G has three differential zeros $x_{4,1}^*$, $x_{4,2}^*$ and $x_{4,3}^*$. It is obvious that the polynomial $F_3(x)$, as the derivative of $G(x)$, can not be negative simultaneously at $x_{4,1}^*, x_{4,2}^*$ and $x_{4,3}^*$. This means that there are at most two x_4^* satisfying the conditions (4). Thus $\mathbf{Q}_T$ has at most 4 centers.

The proof finishes showing that, under condition (4), $\mathbf{Q}_T$ could possess 0, 2 or 4 centers on $\mathbb{S}^2$. This is done in Propositions 5.8, 5.9, and 5.10. $\square$

Proposition 5.8. *Let $a_3=b_1=d_1=0$, $d_0=1$, $d_3=-1$, $b_2=a_1$, $b_3=a_2$, $c_1=2a_1$, and $c_2=2a_2$. Then $\mathbf{Q}_T$ of system (2.12) has no centers on $\mathbb{S}^2$.*

Proof. It is easy to check that in Theorem 5.1 only condition (4) is true. A directly computation leads to $F_3(x) = 3x^2$, $G(x) = x^3 - 1$. Since $G(x)$ has a unique real zero $x_4^* = 1$ which does not verify $F_3(x_4^*) < 0$, we conclude from Theorem 5.1 that $\mathbf{Q}_T$ has no centers on $\mathbb{S}^2$. □

Proposition 5.9. *Let $a_3 = 0$, $b_1 = b_2 - a_1 = b_3 - a_2 = 1$, $c_1 - 2a_1 = d_0 = -1$, $c_2 - 2a_2 = -2$, $d_1 = 3$, $d_2 = -16$, and $d_3 = -43$. Then $\mathbf{Q}_T$ of system (2.12) has exactly two centers on $\mathbb{S}^2$.*

Proof. Obviously, the parameters only satisfy condition (4) of Theorem 5.1. By direct computation we have $F_3(x) = 3x^2 + 12x - 13$ and $G(x) = (x+2)(x+7)(x-3)$. Then $G(x)$ has a unique zero $x_4^* = -2$ which verifies that $F_3(x_4^*) < 0$. Hence by Theorem 5.1, $\mathbf{Q}_T$ has exactly two centers at

$$\Big(\pm\sqrt{(\sqrt{61}-5)/18}, \mp\sqrt{2(\sqrt{61}-5)/9}, -(\sqrt{61}-5)/6\Big). \qquad \square$$

Proposition 5.10. *Let $a_3 = b_1 = d_3 = C_2 = 0$, $d_0 = -5$, $d_1 = 9$, $d_2 = -5$, $b_2 - a_1 = 1$, $c_1 - 2a_1 = -1$, $c_2 = 2a_2$, $b_3 = a_2$. Then $\mathbf{Q}_T$ of system (2.12) has exactly four centers on $\mathbb{S}^2$.*

Proof. By direct computation we have $F_3(x) = -15x^2 + 18x - 4$, $G(x) = -x(5x-4)(x-1)$. $G(x)$ has exactly two zeros $x_4^* = 0, 1$ which verify that $F_3(x_4^*) < 0$. Hence by Theorem 5.1, $\mathbf{Q}_T$ has exactly four centers at $(\pm 1, 0, 0)$ and $(\pm\sqrt{\sqrt{2}-1}, \pm\sqrt{\sqrt{2}-1}, 1-\sqrt{2})$ respectively. □

Proof of Corollary 5.3. Suppose that x_0 is a common zero of $D(x)$ and $G(x)$. By the expression of $D(x)$ we have $x_0^2 = (B_2 + C_1 + (2B_3 + C_2)x_0)/(5a_3)$. Using this formula we can derive by computation that $G(x_0) = \alpha_1 - \beta_1 x_0$. Therefore, if $\beta_1 \neq 0$, then $x_0 = \alpha_1/\beta_1$ and hence condition (5) of Theorem 5.1 is equivalent to condition (1) of Corollary 5.3. If $\beta_1 = \alpha_1 = 0$, then $G(x)$ vanishes whenever $D(x) = 0$. Thus the common zeros of $D(x)$ and $G(x)$ are $(2B_3 + C_2 \pm \sqrt{\Delta_3})/(10a_3)$. Consequently condition (5) of Theorem 5.1 is equivalent to condition (2) of Corollary 5.3. □

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